IMVI **ISSN (p) 2303-4882, ISSN 1840-4383**
Open Mathematical Education Notes
Vol. 4 (2014), 9-18
www.imvibl.org / JOURNALS / IMVI OMEN

ON THE DUALITY OF TRANSITION FROM VISUAL TO SYMBOLIC IN THE TEACHING OF SCHOOL MATHEMATICS

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Abstract

This note explores the efficacy and drawbacks of moving from visual to symbolic in the teaching of school mathematics. It stems from the author's experience working with prospective teachers in different mathematical education courses. In particular, the note demonstrates how the appropriate use of computing technology can overcome possible pitfalls associated with an uncritical use of computer-generated images of mathematical concepts. The importance of conceptual perspective on visual representations and the teacher's role in helping students to develop such perspective are emphasized.

Key words: visualization, computers, conceptual knowledge, mathematics teacher education
ZDM Subject Classification: D30, D40

1. Introduction

Mathematics, with its origins in the study of number and shape, has evolved from concrete activities to abstract concepts by means of argument and computation [4]. The first mathematical problems, as known from history, stemmed from the contexts of counting using "the principle of one-for-one correspondence … without a need for names for numbers" [19, p. 31]. Later, the physical manipulation of objects and visual argumentation regarding the relationship among the objects led to the need for names describing specific properties of numbers. For example, "being ancient even in Plato's time [380 B. C.] … [was the game of] guessing odd or even with respect to the number of coins or other objects held in hand" [22, p. 16] and a geometric term gnomon, resembling a sundial (an instrument that determines the time of the day by the position of the Sun), was used to refer to an odd number because of its double plus one form. Over the centuries, the development of mathematical knowledge evolved by taking into account the primordial nature of concrete objects including geometric shapes over the secondary nature of words and other signs that describe specific combinations and properties of those objects. In the words of Vygotsky, mathematical knowledge had been developed through the transition from dealing with the "first-order symbols … directly denoting objects or actions

…[to] the second order symbolism, which involves the creation of written signs for the spoken symbols of words" [24, p. 115]. Just as the teaching of writing was recommended to "be arranged by shifting the child's activity from drawing things to drawing speech" [24, p. 115], the teaching of mathematics can be arranged as a transition from seeing and acting on concrete objects to describing the visual and the physical through culturally accepted mathematical notation.

To this end, the "we write what we see" (W^4S) principle can be proposed as a didactic motto to be used in the teaching of mathematics. We see differences and similarities when dealing with geometric figures or their images and appreciate different terminology to describe them [16]; we see a relationship (known as the triangle inequality) among the lengths of three straws when trying to construct a triangle that doesn't (or does) allow for such a construction [2]; we see within a numeric table that the sum of two consecutive triangular numbers is the square of the rank of the larger number, an observation used in the 18th century by a Dutch minister of church and mathematics teacher Élie de Joncourt to compute squares and square roots [18]. There are plenty examples of that kind in school mathematics and beyond.

Yet, the W^4S principle works not without reservations. Although the avowal "I see" often confirms understanding, mathematical visualization, as Tall put it, "has served us both well and badly" [23, p. 105]. Therefore, while the focus on visualization is a commonly accepted practice of mathematics teaching and learning [17, 25], especially in the digital era (e.g., [8]), there is a long and sometimes challenging path from seeing things to understanding correctly their mathematical meaning or the absence thereof.

The advent of computers in the classroom has provided great many opportunities for visualizing mathematical concepts using software programs for the construction of graphs, diagrams, geometric shapes, numeric tables, and even step-by-step solutions to complex problems. Notwithstanding, the appropriate use of software is not a simple matter and a teacher has to possess both mathematical and technological skills in order to, whenever possible, provide students with conceptually accurate images of mathematical ideas under study. Likewise, a deep knowledge of mathematics is required to provide infallible visualization in off-computer environments. As Wittmann put it, "The most important thing in teaching is to understand mathematical structures as teaching aids that facilitate learning" (cited in [3, pp. 361-362]). The present note describes some pedagogical ideas grown along the above lines and born in the context of the author's work with prospective teachers of mathematics.

2. W^4S principle and the duality of its affordances

The limitations of uncritically using the W^4S principle become obvious already at the pre-school level. Indeed, we can see that two pineapples are bigger than three plums. As one moves from visual to symbolic, the first and the second kinds of fruit can be associated with the numbers 2 and 3, respectively. But this does not imply that labels attached to the numbers may be dropped leading one to conclude that in the domain of the second order symbolism two is bigger (greater) than three. This is a simple example of how the adage "A picture is worth a thousand words" may be misleading in the absence of conceptual understanding of the dual nature of educational affordances, positive and negative, that a picture provides.

In what follows, the duality of affordances of the W^4S principle in the teaching of mathematics will be discussed. The theory of affordances [10] is frequently used nowadays when talking about teaching mathematics with computers [13, 14]. However, what is true for a computer environment is also true for any learning environment. In general, the more positive affordances an educational environment offers, the fewer negative affordances it presents. At the same time, negative affordances of a particular pedagogical approach are often hidden and an uncritical use of any approach can lead one astray in the learning of mathematics, thereby increasing the effect of hidden didactic challenges. In order to minimize negative affordances of a learning environment, the ability to conceptualize first order symbols created through action is crucial.

Kaijevich & Haapsalo [12] referred to a case when procedure is informed by concept as an educational approach to the teaching of mathematics. Such conceptually informed procedure may include the creation of the first-order symbols toward the end of developing their interpretation through the second order symbolism. Conceptual understanding plays a critical role not only in seeing things in terms of understanding them but creating educationally flawless visual representations of mathematical concepts. In turn, conceptual perspective on visual representations can turn negative affordances of a learning environment into its positive affordances.

Mathematical knowledge develops from action on concrete objects to their formal description through words and/or mathematical notation. So, in the teaching of mathematics one may encourage students at all levels to start doing mathematics from acting on the first order symbols and then, through the appropriate use of the W^4S principle, make a transition to the second order symbolism abstracted from the concreteness of visual representations. Appropriate use of technology can be defined as balancing positive and negative affordances of what technology (which may include more than one digital tool) provides; ideally, maximizing positive affordances and minimizing negative affordances of the tools. Because it is a teacher who "has a critical responsibility in shaping the relation between the computational media and mathematical knowledge" [9, p. 200], courses for prospective teachers of mathematics must provide guidance on how to shape this relation starting from the very basic examples of using the W^4S principle. The same is true regarding the relation between non-digital teaching aids and mathematical concepts they are designed to support.

3. W^4S principle in teaching primary school mathematics

As was mentioned above, even in rather simple situations, the W^4S principle might give misleading results in the absence of conceptual understanding. In fact, conceptual understanding can be fostered through the use of counterexamples: a combination of two pineapples and three plums can be used to develop the appreciation of the concept of unit in modeling the relationship between (or among) whole numbers. Quantitatively, those objects (pineapples and plums) are not comparable due to different units they comprise. Likewise, without recourse to the notion of experimental probability one cannot easily respond to the question about chances of randomly picking a specific fruit from a basket with the five fruits. That is, seeing a picture does not mean that, in the absence of conceptual understanding, one can describe it quantitatively in a correct way.

In order to develop such understanding of comparing quantities, a teacher can give students linking cubes (non-digital technology) of the same size to construct towers 2-cube tall and 3-cube tall and then ask several question such as:

- What do you see? Which tower is taller?
- What can be said about the numbers 2 and 3? Which one is bigger?

In the digital era, comparing quantities can be facilitated (and conceptually enhanced) by using a computer program, such as *The Geometer's Sketchpad* (created by Nicholas Jackiw in the late 1980s and commonly used in the schools in North America) which can help one to construct towers out of same size squares. Here, the main idea is to have students construct squares all the same size by appropriately using construction features of the program. Note that the W^4S principle works the same way for the fruit and the squares when the total number of objects has to be determined: in both contexts one can see without any reservation that $2+3=5$. Conceptual difficulties with addition begin with the introduction of a base system that can be overcome through the appropriate use of the W^4S principle in the context of base-ten (or multi-base) blocks [2].

The Geometer's Sketchpad can also be used to construct a simple program for comparing fractions through fraction circles integrating conceptual understanding into a computer-mediated action. Consider the task of constructing the fraction circles 1/2, 1/4, 1/6, and 1/12 (by defining the location of their center and the length of the radius), arranging them from the least to the greatest, and finding their sum. Both operations should be presented in iconic and numeric forms.

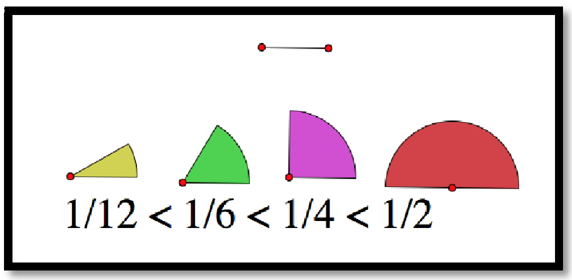

Figure 1. We write what we see.

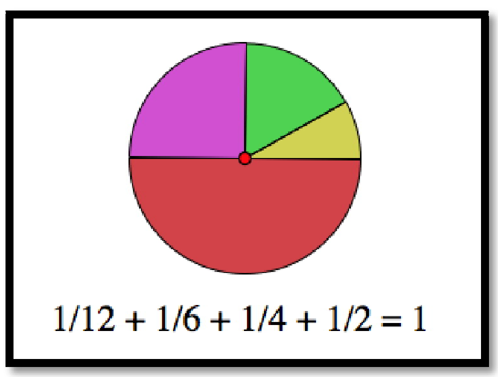

Figure 2. Addition automatically requires equal radii.

The main focus of this task deals with a frequently overlooked (or taken for granted) fact that fractions may be compared only when the same unit is their point of reference. In terms of icons (the first order symbols), the fraction circles 1/2, 1/4, 1/6, and 1/12 are four different sectors cut of the same whole circle. In terms of the second order symbols, these three fractions are parts of the same unit. This idea is hidden in the construction of a fraction circle when one defines its radius. As shown in Figure 1, the four fraction circles have the same radius pictured at the top of the sketch. Therefore, they may be compared both as the first order symbols (icons) and the second order symbols (numbers). Finally, when adding the four fractions (Figure 2), their radii can be adjusted through an action that brings about a new picture, the description of which in the domain of second order symbols is then made in terms of a numeric equation that becomes independent of any context. Here, the concept of same unit is implicitly embedded in the construction of the very unit using its parts. That is, unlike comparing numbers, adding them has to be carried out correctly already in the domain of the first order symbols. At the same time, as shown in Figure 3, when adding fractions representing different units one is apparently unable to describe the sum at the level of the second order symbolism.

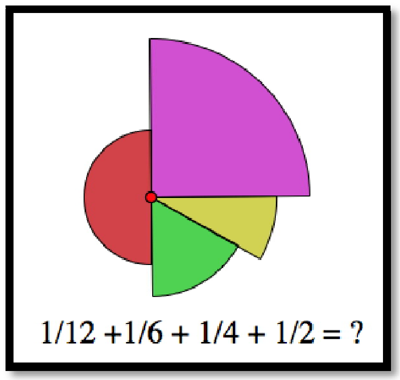

Figure 3. The sum does not make sense.

The above use of fraction circles emphasizes the fact that when one carries out arithmetical operations with fractions, it is assumed, though tacitly, that they are fractional parts of the same unit. In that way, using the computer program not only "require[s] the user to *describe* intended relationships" [11, p. 365, italics in the original] but force them to integrate meaning with the required construction of the objects of visualization. Therefore, a conceptual flaw occurring at the action level (constructing fraction circles with different radii) might result in the erroneous ordering of the objects based on their size, followed by an incorrect symbolic description through an uncritical use of the W^4S principle. In the case of adding such fraction circles (Figure 3), no symbolic description of the sum can be offered. So, only the equal radii construction of fraction circles allows for their comparison, otherwise the comparison of visual images is meaningless. At the same time, the operation of adding fraction circles representing different units as the first order symbols does not yield *any* result at all at the level of the second order symbolism. It is not surprising that, conceptually, inequalities are considered being more sophisticated entities of mathematics than equalities, although, in the case of small numbers, the former do not require an operation while the latter do.

4. On the deficiency of drawing: comparing fractions using area model

As a more complicated arithmetical example, consider the case of comparing the fractions $\frac{3}{5}$ and $\frac{4}{7}$. Which one is bigger? Figure 4 shows a one-dimensional method of the comparison of fractions using the so-called area model for fractions. Whereas this method works well for comparing unit fractions such as $\frac{1}{2}$ and $\frac{1}{3}$, for non-unit fractions that are sufficiently close to each other, this method stops working because of its dependency on the accuracy of drawing. As shown in Figure 4, looking at the representation of the fractions from left to right one sees what can be described symbolically as $\frac{3}{5} < \frac{4}{7}$; looking at the same representation of the fractions from right to left one sees the opposite relation, $\frac{3}{5} > \frac{4}{7}$.

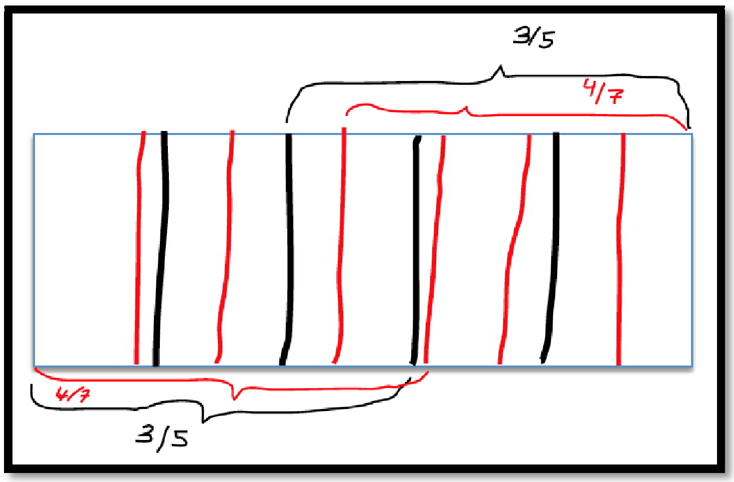

Figure 4. Visualization depending on accuracy in drawing is contradictory.

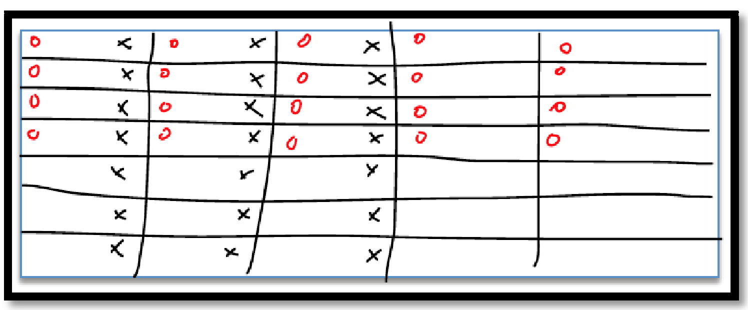

Figure 5. Comparison of fractions through counting marks.

Rather than discouraging the use of the W^{4}S principle, this example shows the deficiency of the one-dimensional representation of the comparison of fractions and thereby, it motivates a two-dimensional method shown in Figure 5. So, the negative affordances of the one-dimensional representation were used as a counterexample in comparison of fractions and motivated the introduction of a two-dimensional comparison of fractions using area model and the appreciation of the positive affordances of the method. That is, the misleading diagram of Figure 4, serving as a counterexample to the uncritical use of the W^{4}S principle, plays an important role in fostering conceptual understanding of fractions as the second order symbols. At a higher level, one can recognize in the diagram of Figure 5 the concepts of the common denominator of two fractions (a cell of the grid) and the product of the fractions (the overlap of two different marks), something that is definitely missing in Figure 4. Such recognition of the basic concepts of arithmetic is due to the power of the W^{4}S principle, which works well within a flawless learning environment of the first order symbols.

5. Using the W^{4}S principle at the secondary level

How many solutions does the equation $\tan x = x$ have? According to [9], when secondary school students see the image of the graphs of the functions $y = x$ and $y = \tan x$ on a computer screen, their answer, a finite number of solutions, is affected by what they see within the range $[x, y] = [-10, 10; -20, 20]$. Indeed, such a question cannot be answered in a computer graphics environment without an appropriate application of the W^{4}S principle. Even more, the students' confusion was with what they saw in the neighborhood of the origin: the two graphs didn't look like having the origin as the only point of intersection. It appears that a teacher has to be aware of such instances of the misleading use of the W^{4}S principle and utilize them as counterexamples in order to motivate alternative approaches to graphing. So, the question about the number of solutions of the equation $\tan x = x$ may be considered a TITE problem [1]; that is, a problem that is both technology immune (TI) and technology enabled (TE). In other words, such a problem cannot be solved by technology alone; yet, technology plays a critical role in its resolution. In particular, one cannot correctly answer the question about the number of roots of the equation without understanding how mathematical concepts can be integrated with the appropriate use of technology. What we want to demonstrate is the infinite number of the roots of this equation rather that the infinite number of the points of intersection of the graphs $y = \tan x$ and $y = x$. So, an emphasis has to be on what is happening on the *x*-axis (where the roots are located) and not in the (*x*, *y*)-plane.

Furthermore, the straightforward graphing of the two functions does not show their mutual behavior using two dimensions when both *x* and *y* have to be seen for sufficiently large values. With this in mind, one can use the *Graphing Calculator* [5], software capable of graphing relations from any two-variable equations and inequalities. By using this tool to graph in the $(x, y)-$plane a system of the relations $x = \tan x, |y| < \varepsilon, x > 0$, where ε is a sufficiently small positive number ($\varepsilon = 0.03$ in Figure 6), one can clearly see much more than when the graphs $y = \tan x$ and $y = x$ are constructed. Seeing the points of the *x*-axis where *y* = 0 requires just the proximity to the *x*-axis and the computer graphing environment selected, allows one to move along the *x*-axis for sufficiently large values of *x*. This is an example of how, using conceptual understanding, one can turn negative affordances of the learning environment into its positive affordances. That is, positive affordances of the *Graphing Calculator* can be revealed through conceptual understanding of how "software can embody a mathematical definition" [7, p. 132] allowing one to see the roots of a one-variable equation in the neighborhood of the *x*-axis for sufficiently large *x* (Figure 6).

Seeing the *Graphing Calculator* as an instrument and using mathematics to transform it beyond the straightforward graphing of the functions involved, represents one of the components of the instrumental genesis [8]. Infinity is an abstraction and its visualization can only be approximate. The idea of using the *Graphing Calculator* to demonstrate the phenomenon of infinite number of solutions to an equation becomes an agency for mathematical activities through which one learns to construct a system of relations in two variables so that its graph enables one to visualize the phenomenon of the infinite number of solutions of an equation involving a circular function.

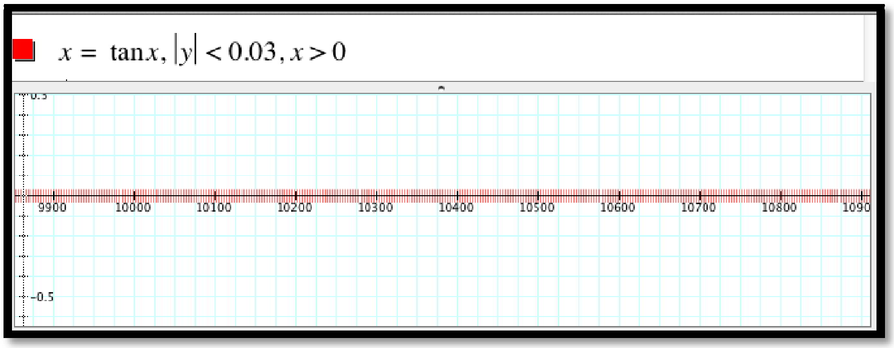

Figure 6. The equation $\tan x = x$ continues having roots for sufficiently large values of *x*.

6. Conclusion

This note focused mostly on the cases when an uncritical use of the W^{4}S (we write what we see) principle in the teaching and learning of school mathematics can lead to incorrect mathematical interpretation of visual images provided by commonly used learning environments. There are many concepts outside school mathematics for which the W^{4}S principle works nicely with or without help of computer graphing. These include Bolzano's Theorem (regarding a continuous function that vanishes within a certain interval at the endpoints of which it assumes values of opposite signs) and its generalization known as Intermediate Value Theorem; Rolle's Theorem (regarding the vanishing derivative of a continuous function which itself vanishes at least twice) and its generalization known as Mean-Value Theorem. These concepts, however, do not involve the infinity. A classic example involving infinity is the early uses of a computer in deciding the convergence of the harmonic series [15] or other slowly diverging series. For instance, if one uses a spreadsheet to fund the sum of some first 30 million terms of the harmonic series, the sum is only about 18. A similar example was provided in the context of using a graphic calculator in finding limit of the logarithmic function [9].

In such conceptually complex milieu, it is the teacher's responsibility to demonstrate to students both flawless and flawed visual representations of a mathematical phenomenon. In some cases, a flawed visual representation (like comparison of two fractions shown in Figure 4) may serve as a motivation for a more sophisticated, yet flawless and conceptually rich representation (like the one shown in Figure 5). In other cases, a flawless representation (comparing same size squares to decide the relationship between numbers) may be provided first so that its flawed analog (different kinds of fruit) can be shown afterwards to reveal and underline the main idea imbedded into the former representation.

The above recommendations apply to the teaching of prospective teachers as well as their students, because, as is well known, teachers tend to teach as they have been taught [20] and such a tendency may reflect their whole mathematics learning background. As the author's experience working with teacher candidates in different mathematics education courses suggests, the W^{4}S principle works well as a learning concept in the classroom. An appreciation of the dual affordances of the principle by teacher candidates is due to the courses' strong focus on the classroom pedagogy at the level of mathematics itself for a good command of pedagogical content knowledge [21] is currently considered as the basis for students' progress in the learning of mathematics [6]. In the digital era, this progress can be accelerated by the appropriate use of technology.

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Received by editor 12.09.2014. Available online 03.11.2014.