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EDUCATING TEACHERS TO POSE MATHEMATICAL PROBLEMS IN THE DIGITAL AGE: TOWARD ALTERNATIVE WAYS OF CURRICULUM DESIGN

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Abstract

This paper shows how by integrating technology with problem-posing activities prospective teachers of mathematics can learn new ways of creating curriculum materials that address several pedagogical issues associated with the advent of modern digital tools into the classroom. The paper describes and analyses problems posed by the teachers for different populations of students under the umbrella of two interrelated notions: the didactical coherence of a problem and the technology-immune/technology-enabled problem. Digital tools discussed in the paper include an electronic spreadsheet, *Wolfram Alpha*, and *Maple*. It is shown that the tools cannot be directly utilized for posing mathematical problems and that such utilization requires one's appreciation of their hidden educational capabilities and awareness of instructional pitfalls.

Key words: problem posing, problem solving, teacher education, mathematics, technology

ZDM Subject Classification: **D30, D40**

1. Introduction

The objective of this paper is to demonstrate how the appropriate use of modern tools of technology can be integrated with problem posing activities in a broad context of mathematics education, including the preparation of teachers. One goal of such integration is to provide context for teacher candidates' learning to use technology through posing grade appropriate problems. Another goal is to provide teacher candidates with knowledge that allows for new ways of diversifying curriculum materials that address the advent of modern technology tools into the mathematics classroom.

It has been more than two decades since the importance of learning problem posing in the digital age was expressed by the National Council of Teachers of Mathematics [50], the major professional organization of mathematics educators throughout the United States and Canada, through the affirmation of the educational promise of activities that encourage students' creative work leading to the ownership of curriculum materials. This belief in the power of technology as a means of pedagogy was enhanced theoretically through the technological pedagogical content knowledge (TPCK) framework [9, 47, 53] developed at the confluence of Shulman's [63] notion of pedagogical content knowledge (PCK) and that of Maddux's [46] Type I/Type II technology applications. The PCK notion pointed to the fact that "questions about the *content* of the lessons taught, the questions asked, and the explanations offered" [63, p. 8, italics in the original] are unjustifiably missing from research on teaching. Similarly, the Type I/Type II notion pointed to the fact that because "so many educators seem to be simply taking it for granted that microcomputers are desirable, ... such uncritical and unrealistic optimism is doomed from the start and [without good research] could result in a backlash reaction against educational computing" [46, p. 36]. Thus, at the confluence of the calls for subject matter knowledge and the appropriate use of technology, the notion of TPCK was developed. This paper suggests that one of the directions to guide teacher candidates "in expanding their understanding of the interactions of the knowledge of technology and the knowledge of their subject area" [53, p. 520] in order "to transcend the passive learner role and to take control of their learning" [47, 2006, p. 1035] and investigate "how to assess learning-in-progress as teachers advance from novice to expert thinking about designing instruction with technology" [9, 2009, p. 162] is a technology-enabled mathematical problem posing. That is, posing problems with technology develops teachers' professional competence, motivates active learning, encourages reflection, and fosters metacognitive skills [77].

Problem posing by the learners of mathematics, in general, has been on the mathematics education agenda for a long time appearing in different didactic forms as a way of enriching one's learning experience through investigating mathematical ideas, exploring conjectures, and solving worthwhile problems. In the digital age, teacher candidates can learn using this experience in formulating a variety of mathematical problems that the appropriate use of technology affords. In the context of this paper, the word *technology* refers to commonly available software tools including an electronic spreadsheet and various computer algebra systems. The word *appropriate* refers to the fact that the tools of technology cannot be directly utilized for mathematical problem posing. Rather, such utilization requires one's appreciation of their hidden educational capabilities as well as instructional pitfalls.

This paper shows how the TPCK concept, when mapped onto the domain of mathematics education, can elucidate a number of the concepts associated with problem posing in the technological paradigm. One such concept, originally introduced in [6] to help teacher candidates become better problem posers in the digital age, is that of didactic coherence of a problem. Another concept, described in detail elsewhere [3], is that of technology-immune/technology-enabled (TITE) problem. This latter concept emerged from the observation that using modern technology tools capable of symbolic computations (e.g., a web-based computational engine *Wolfram Alpha*) can inadvertently put barriers in the way of developing students' creative thinking by reducing mathematical problem solving to merely button pushing on a computer. This, in turn, opens a new window on the alternative way of posing problems in the digital age. Therefore, the ability of teacher candidates to formulate and solve problems using technology has great potential to enhance the field of curriculum design through offering flexible alternatives to conventional educational materials.

The ideas of this paper stem from the author's work with American and Canadian teacher candidates (hereafter referred to as teachers) in different technology rich mathematics education courses. The context of problem-posing activities shared by the author includes pre-service teacher education courses and activities with elementary students administered by teachers as part of their fieldwork. The paper shows that a teacher's success with technology-enabled problem posing in the context of fostering creative thinking in the students of mathematics requires experience with mathematical modeling and problem solving. It demonstrates that technology-enabled problem posing provides teachers with research-like experience in mathematics pedagogy and then leads them to have an ownership of their learning of grade-appropriate extensions of the traditional mathematical content.

2. Mathematical problem posing and teacher education

Mathematics, the basic entities of which comprise number and shape, begins with posing problems. It evolves from concrete ideas to abstract concepts by means of argument and computation [8]. Many problems already found in the mathematics of antiquity were first posed in a real life context to allow for physical manipulation and cognitive experimentation. Formulation of such problems was a reflection on the very context and their solution was facilitated by this context. As mentioned in [38, p. 4], "activities are much more effective than conversations in provoking problems". Already at the early age, activities from which problems arise serve as means of stimulating one's intellectual growth. Words, as units of conversation, are used to describe manipulable physical reality. In that way, as mathematical knowledge develops from action to its formal description, problem posing as an educational project takes into account the primordial nature of concrete things (the first order symbols) over words (the second order symbols) that describe those things [74].

Consider the following problem from the Babylonian mathematics (2000-1600 BC) in which geometry serves as a frame of reference in facilitating argument and computation: "*Length, width. I have multiplied length and width, thus obtaining the area. Then I added to the area the excess of the length over the width ... 183 was the result ... Required length, width and area*" [72, p. 63]. In this formulation, geometric entities are used as the first order symbols from which computations expressed through the second order symbols, to allow for argument, follow. This approach to provoking a problem, avoiding the direct use of abstract symbols, makes mathematical computations more concrete. The historical and psychological perspectives on problem posing and solving can inform the current pedagogy of mathematics teacher education by encouraging the formulation of problems "with the learner in mind" [70, p. 416].

Teaching "the craft of task design" [21, p. 65], that is, teaching to pose problems, has long been recognized as an important pedagogical tool in work with different populations of mathematics students [12-14, 29, 30, 33, 40, 41, 59, 61, 69, 75]. Influenced by pioneering research carried out in this area, the National Council of Teachers of Mathematics has referred to problem posing as "an activity that is at the heart of doing mathematics" [49, p. 138]. Moreover, drawing on the technological advances of the digital age, the Council suggested "computer programs can engage students in posing and solving problems" [49, p. 76]. Towards this end, already in the early 1990s, the importance of problem posing for the development of mathematical thinking with the roots in Socratic seminars which encouraged participants to generate questions [16] and the growth of educational applications of computers rooted in the belief that "mechanical aids are possible which ... would leave the teacher more free for her most important work, for developing in her pupils fine enthusiasms, clear thinking and high ideas" [60, p. 376], have been linked together in the standards for teachers: *"*technology may be used to enhance and extend mathematics learning and teaching … in the areas of problem posing and problem solving … [allowing] students to design their own explorations and create their own mathematics*"* [50, p. 134]. The first most commonly known examples of using technology for posing mathematical problems included the development of conjectures in dynamic geometry environments (e.g., [43, 76]), once again, using the concreteness of geometry to facilitate argument and computation. Nowadays, the appropriate use of spreadsheets and computer algebra systems enable other areas of pre-college mathematics to be explored from a problem-posing perspective.

For example, at the elementary level, by using a spreadsheet, one can turn a routine arithmetical problem of finding a sum of consecutive natural numbers into a challenging mathematical investigation. Such a transition from a routine to a challenge is based on the ease of formulating "what if" questions in a spreadsheet environment the resolution of which requires a move from particular to general and the use of ostensive definitions in dealing with such a move. In other words, as noted in [40], the availability of technology-enabled variation of the conceptual and syntactic structure of an existing problem statement provides a source of new problems.

Likewise, at the secondary level, using a spreadsheet in numeric modeling of a Diophantine equation (a polynomial equation in several unknowns with integer coefficients to be solved in integers) in two or three unknowns can facilitate posing a multitude of algebraic word problems leading to both linear and non-linear equations with friendly solutions. Computer algebra systems such as increasingly used in mathematics education *Wolfram Alpha* (see, e.g., [26, 51, 68]), *Maple* [15], or the *Graphing Calculator 4.0* [10] also provide medium for posing grade-appropriate problems. However, the capability of these systems to carry out intricate symbolic computations opened both new educational opportunities and didactic challenges for mathematics educators. In terms of the theory of affordances [34] frequently used nowadays when talking about technology [9, 39, 45], the more positive affordances a tool offers, the fewer is the number of negative affordances it presents. However, negative affordances of a computational tool are often hidden and the use of technology in work with secondary school students is not a simple matter. For instance, a computer can now understand even a not formally structured question and, in response, offer several options for a student to select the correct answer. Also, many traditionally formulated problems can be solved unambiguously by software with a minimal contribution by a student.

Thus, the appropriate use of technology in mathematics education can be conceptualized as a process that maximizes positive and minimizes negative affordances of a computational learning environment.

Integrating technology in mathematics teacher education courses provides teachers with research-like experiences in the context of posing problems through analyzing numeric, symbolic, and graphic results generated by software. Often, especially in the context of spreadsheets, numeric modeling requires the development of computer models using mathematical machinery. In such a case, technology firstly provides an agency for mathematical activities by the teachers, secondly technology starts consuming the activities through exploring numerically similar models, and, finally, technology turns into an amplifier of the activities [2].

While there are many studies concerning teacher education and mathematical problem posing (e.g., [22, 25, 35, 66]), until recently they did not deal with the use of technology. The following sections of the paper describe problem-posing activities that the above-mentioned computer programs naturally afford. The paper shares the author's analysis of problems posed by teachers using these tools of technology. In the context of courses for elementary teachers it will be shown how one can avoid using algebra otherwise required for posing grade-appropriate problems with friendly data. In the context of courses for secondary (mathematics) teachers, it will be demonstrated how the appropriate use of the tools allows one to use technology as an agent of mathematical activities leading to the creation of problems being unsusceptible to computer-based straightforward numeric and symbolic computations as a method of finding an answer [3].

3. The didactic complexity of problem posing with technology

The use of technology in problem posing can be characterized as a cultural support of teachers' ability to design new curriculum materials for a mathematics classroom. It is cultural because one is encouraged to use the tools of technology developed by advanced members of a technological culture for various practical and scholastic purposes and, in the case of a spreadsheet, originally developed for non-educational purposes, retrofitted by others for mathematics education. It is support in a sense that, in the specific context of problem posing, teachers learn how to put to work the power of technology in order to formulate a grade-appropriate problem. In some cases, technology generates solutions to a problem that is about to be posed and the task for a teacher is to recognize this relation between the two sides of the same coin, problem posing and problem solving. This implies that problem posing and problem solving are inherently linked to each other through the use of technology.

Seeing technology as a cultural support of mathematical problem posing, one should note that the availability of powerful computational tools does not guarantee their appropriate application unless one examines the effects of support in the context of using the tools [18]. To support this position, the ideas of this paper, to a large extent, resulted from the author's analysis of problem-posing activities by teachers. As part of the activities, teachers were expected to critically reflect on their own problems and discuss the role of computational environments they used to develop the problems.

However, in order for technology to have a positive effect on problem posing, one should not only know how to use it but, more importantly, how to interpret the results that a technology tool generates. This interpretation requires understanding of what may be called the didactical coherence of a problem. This notion can be applied to a non-digital problem posing as well. In the following sub-sections the detailed description of these ideas is presented.

3.1. Numerical coherence in problem posing

Numerical coherence of a problem refers to its formal solvability within a given number system. In turn, a number system involved in a problem depends on a grade level. For example, a problem of dividing two cakes between three children is not numerically coherent for the first graders who are familiar with whole numbers only. Yet, they can physically perform this division (if we look for fair sharing). Why does one need a number to describe the result that can be physically carried out? In other words, why does one need to move from symbolism of the first order (manipulating concrete things) to that of the second order (dealing with abstractions that describe those things)? One needs the second order symbol (a number) when dealing with the first order symbols (physical division of the cakes) is not the end of the story and the result, for instance, has to be quantitatively compared with a similar situation. At the same time, two bicycles cannot be divided among three children, either numerically or physically.

This shows the primordial role of physical action over its symbolic description. Another example of numerically incoherent problem is that of constructing a triangle with the side lengths 2, 3, and 6 linear units. Here, however, the change of number system does not affect the problem's solvability. Rather, to make this problem numerically coherent one has to choose three side lengths satisfying the classic triangle inequality – the sum of any two sides of a triangle is greater than the third side.

Numerical coherence of a problem can be obvious, like finding the sum of the first *n* natural numbers – whatever the answer is, it is a natural number because natural numbers are closed under the operation of addition. This is what the modern day educational document in the United States, Common Core State Standards [20], referred to as "the use of structure" by students as they learn to develop formal reasoning skills. By the same token, numerical coherence can be hidden – like partitioning a natural number into a sum of consecutive natural numbers. A solvability of this partitioning problem depends on a number that has to be expressed as a sum of consecutive natural numbers because not all natural numbers afford such a representation. So, numerical coherence of a problem is not a simple matter.

Teachers often seem to miss the point that problem posing "is a platform from which further development proceeds" [23, p. 23] and don't move beyond posing a problem. Typically, earlier research on problem posing supported this point of view [64, 65]. Some studies, however, suggested that the two activities – posing and solving a problem – are in the relation of dichotomy. One such study [22] indicated that knowing how to solve a mathematical problem is not enough to be able to pose a problem. This view may be due to the fact that just like many young children traditionally see their role at school as an engagement in answering, not asking questions [71], many teacher candidates believe that problem posing activity ends with posing a problem rather than being followed by its problem-solving phase. Nonetheless, when a spreadsheet is used to pose a problem, the answer is there as a computational setting is designed to generate the answer. In order to recognize the answer within a computational environment, one has to interpret correctly the results of spreadsheet modeling. As an illustration, consider a problem posed by one of the (elementary) teachers using a spreadsheet designed to numerically model a linear Diophantine equation in three unknowns.

Problem 1. *Casey goes to the mall for her birthday! She goes to her favorite toy store where everything is on sale for \$15, \$10, or \$5. If Casey has a total of \$30, how many different ways can she spend it in the store?*

The spreadsheet of Figure 1 shows that there are seven ways to spend money in the store: the software counts the number of non-empty cells in the range D5:K9 and displays this number in cell A8. For example, the triple of the cells E4, C6, and E6 gives the following representation: $30 = 1{\cdot}15 + 1{\cdot}10 + 1{\cdot}5$. One can say that Problem 1, the mathematical model of which is the equation $15x + 10y + 5z = 30$, is numerically coherent and the triple $(x, y, z) = (1, 1, 1)$ is one of its solutions. This is how the teacher who posed the problem explained the concept of numerical coherence.

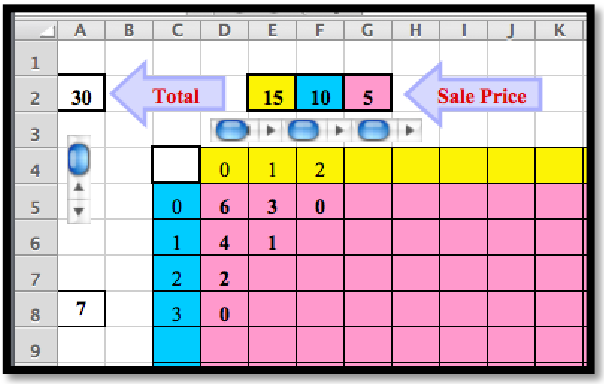

Figure 1. Spreadsheet solution to Problem 1.

My problem is definitely numerically coherent. \$30 is a multiple of all the sale price amounts that I gave (\$15, \$10, and \$5). If I had changed the amount to something different, say \$27, then my *problem would not be numerically coherent. It is very important that teachers review any problem they create and make sure that they are numerically coherent*.

Indeed, if one sets the content of cell A2 at 27, cell A8 would display zero reflecting the absence of numbers (the amounts of $5 toys) in the range D5:K9. In other words, the equation $15x + 10y + 5z = 27$ does not have integer solutions. The teacher also attempted explaining how the spreadsheet of Figure 1 works so that one can follow the tool in solving the equation. In fact, students for whom this problem is designed are not expected to use a spreadsheet. Rather, the students are expected to either use trial and error or reason systematically. The latter type of reasoning is used by the teacher in her analysis of how the modeling data of Figure 2 corresponds to a possible paper and pencil solution.

... the first combination was if Casey bought two toys at $15 each: $15+$15; the spreadsheet displayed that. If I broke one of the $15 into $10 and $5, I would have a second combination: $15+$10+$5; the spreadsheet showed that Casey could buy one toy at each of these price levels. If I broke up the $10 into two five dollars, I would have my third combination: $15+$5+$5+$5. . . . [Finally,] I then split up the last $10 into five dollars and got my last combination: $5+$5+$5+$5+$5+$5; the spreadsheet showed that the [sic.] Casey could buy six toys for five dollars each.

In her analysis, the teacher implicitly highlighted the idea of reduction (of a problem with three unknowns) to a simpler problem (with two unknowns), something that can be used when technology is not a part of the learning environment. One can see how one of the most powerful problem-solving techniques in mathematics as well as in mathematical didactics – recall the advice “solve a simpler problem” [58, p. 114] – was incorporated into the very design of the spreadsheet of Figure 1. Students can (and should be taught to) use this idea in their paper-and-pencil work on Problem 1.

3.2. Contextual coherence in problem posing

One of the essential characteristics of mathematics is that its models are capable of describing diverse contexts within which multiple phenomena can be observed. In other words, the same second order symbol can describe different first order symbols. Although originally, a model stems from a single context, the mind constantly searches for other contexts that match a mathematical model which successfully describes a single context. The ability to associate a mathematical model with a context and, vice versa, to match a context with a model, that is, to move freely from one type of symbolism to another, develops through problem posing. Context is very important for understanding mathematics. As was mentioned above, it can be a critical tool in dealing with abstractions. Yet, in order to understand context, one needs to possess social competence, which may vary across different cultures. This requires that the formulation of a problem is consistent with cultural background and social competence of a mathematics classroom that nowadays consists of culturally amalgamated groups of pupils. This brings another type of coherence of a problem.

Namely, *contextual coherence* of a problem comes into play when its solution has to be interpreted in terms of a context within which problem posing occurs. Besides the need to understand the context of a problem statement, it requires one’s appreciation of hidden assumptions grounded into one’s real-life experience and cultural background. As the teacher, justifying contextual coherence of Problem 1 (she posed for a grade three classroom), put it:

My problem deals with birthdays and going to a toy store to buy toys. If the problem was for seventh or eighth grade, I could switch it to birthdays and going to a game shop. Also, I expect that my students all know what dollar bills we have in our country; however, I am aware that if I have a student from a different culture then they may not know how to make the different combinations that add up to $30.

Note that young children can ask a variety of non-mathematical questions about Problem 1 such as: How old is Casey? Who gave her money? What kinds of toys were in the store? Although questions of that type “are essentially requests for information rather than for explanation” [71, p. 277], teachers’ answers to those questions can help the children better understand context in order not to ignore relevant or use irrelevant information. Generally speaking, contextual coherence of a problem is a variable attribute. Just as without the mastery of base-ten system – a cultural tool designed to support one's counting abilities – one cannot understand the meaning of a multi-digit number, without the mastery of other base systems one cannot master base ten at the concept level [73]. This is consistent with Cobb's [17] position that the learning of arithmetic involves the mastery of the numeration system as a cultural tool. In formulating problems in context [62], one has to be sensitive to this issue, as errors that students make when solving problems may result from their own way of understanding context. An example of interplay between context and pedagogy is given through the discussion of the third type of coherence of a problem associated with pedagogical issues.

3.3. Pedagogical coherence in problem posing

Pedagogical coherence of a problem includes such issues as attention to students' on-task behavior, the absence of (or minimizing) extraneous data, the level of syntactic complexity [65], semantic/contextual clarity, grade appropriateness, and a method of solution expected. To explain the idea, following is an example of a pedagogically incoherent problem for a second grade student; a problem that manifests semantic ambiguity within a familiar context. A teacher, who, during her field experience, was assisting the student in his struggle with the problem, described this ambiguity in a weekly journal as follows.

The problem posed the following scenario: "15 students went out for recess. 9 students did not play soccer. How many did play?" Immediately, the student concluded the answer to be 15. I wondered if he had not realized he needed to use subtraction. To respond, I read the question again. However, after the second reading, the student still saw no other possibility than 15. Then he made the comment, "It has to be 15, right? Because 15 went out for recess." As he went on to repeat the comment twice more, I finally realized the student's confusion: he saw the statement about not playing soccer as irrelevant. It was only after I clarified to the student that the question was asking how many played soccer that he understood; within seconds of this revelation, he deduced he needed to use subtraction and came to the correct solution: 6 students.

One can see that syntactic ambiguity of the problem did not allow the student to connect the numbers 15 and 9. Whereas the problem, talking about recess and soccer, was definitely contextually coherent for a second grader, it was not pedagogically coherent for him requiring additional clarification of the whole-part relationship between the numbers. It is only after the teacher's intervention that the student realized that the statement "9 did not play soccer" meant 9 is a part of 15 and, consequently, recognized in the situation a whole-part subtraction problem. This episode underscores the importance of the teachers' shift in focus from their own teaching skills and abilities to what students are learning as a result of this teaching [53]. The notion of pedagogical coherence of a problem does enable such a shift.

In the context of posing problems with technology, pedagogical coherence sometimes runs into conflict with an open-ended approach in the teaching of mathematics. The effectiveness of using this approach has been repeatedly emphasized in mathematics education research [11, 42, 54, 56, 78]. In particular, the use of problems with more than one correct answer is emphasized in open-ended pedagogical situations. However, "quite a few" numbers have the property of being greater than one. What do we want students to learn when offering them a problem with many answers? Do we want them to find all the answers? The ease of generating multiple answers to a single question in a computer environment turns a positive affordance of the learning environment into a negative affordance of educational computing affecting pedagogical coherence of a problem. Consider another problem posed by one of the teachers with the help of a technology.

Problem 2. *In how many ways can one make 50 cents out of pennies, nickels, and dimes?*

Using a spreadsheet like the one shown in Figure 1, one immediately gets the answer – there are 36 ways to change 50 cents into dimes, nickels, and pennies. Whereas the modern pedagogical dictum "single question – multiple answers" helps young children to appreciate the concept of multiplicity of answers in a mathematical problem, posing a question with 36 answers does not make much sense.

Teachers are often confused who uses technology, when, and why it is used. In the context of teacher preparation, it is only teachers who are using technology for posing problems. Yet, their students will be working on a task like Problem 2 in a pencil-and-paper environment. The teacher who posed Problem 2 correctly mentioned this fact through the following statement:

I would not expect young children to find all of the solutions to my problem. It would take a very long time to solve the problem with the numbers provided to them.

The teacher, however, goes on to make a conflicting remark regarding his problem's pedagogical coherence:

I believe my problem is pedagogically coherent because I feel the students will be able to find most of my 36 solutions. It should be fun for students to click around and experiment with different solutions.

It appears that the teacher was too excited about his problem when used the word *most*, meaning, perhaps, *some* (solutions). Notwithstanding, the idea of an experiment in the context of using a spreadsheet is a good one because it points to the fact that one can effectively revisit pedagogically incoherent problem in order to formulate a problem with the single answer. Towards this end, one can be asked to use a problem with a large number of computer-generated solutions (answers) in order to formulate a problem with the single answer. Thus, selecting a particular solution may be a way of turning a pedagogically incoherent problem into a coherent one. After a classroom discussion of what such experimentation with a pedagogically incoherent problem of that type may entail, another teacher has formulated a problem with the single solution by correctly interpreting data generated by a spreadsheet.

Problem 3: *Carol spent exactly 50 cents buying some 10-cent stamps and as many 5-cent stamps as 1-cent stamps. How many stamps of each denomination did she buy?*

In posing this problem, she noted that one of the ways to partition 50 into the summands 10, 5, and 1 is through the equality $50 = 2 \cdot 10 + 5 \cdot 5 + 5 \cdot 1$. Reflecting on the posed problem she wrote:

In my classroom, I will use this question to help the students to begin to form and solve a system of algebraic equations. [Indeed, such a system has the form $10x + 5y + z = 50$, $z = y$, leading to the equation $5x + 3y = 25$ with the single solution $x = 2$, $y = 5$]. These types of questions also lead to a discussion of LCM [least common multiple] and GCF [greatest common factor], changing the total amounts and the amounts of the price of the stamps.

The teacher didn't explain how the concepts of LCM and GCF could be connected to Problem 3. The author can only guess what her thinking had been: if the largest denomination were 9-cent, then the problem is not numerically coherent. Indeed, in the resulting equation, $9x + 6y = 50$, whereas GCF(9, 6) = 3 we have GCF(3, 50) = 1. (Alternatively, LCM(50, 3) = $50 \cdot 3$, indicating that 50 is not divisible by 3). This slight modification is different from Problem 3 where in the resulting equation, $5x + 3y = 25$, we have GCF(5, 3) = 1 and GCF(1, 25) = 1. In her unexplained comment, the teacher naturally touched upon Dewey's (1938) pedagogical construct of collateral learning discussed in the forthcoming sections.

To conclude this section, note that pedagogical coherence of a problem depends on the expected method of solution. Often, as students learn using more and more sophisticated mathematical tools, a numerically and pedagogically incoherent problem for a lower-grade level (like dividing two cakes among three children) becomes numerically and pedagogically coherent for a higher-grade level. The opposite relationship can be observed as well: a pedagogically coherent problem for a lower-grade level may become pedagogically incoherent for a higher-grade level. For example, whereas for a six-year-old pupil (who uses concrete materials – the first order symbolism – as means of problem solving) the tasks of arranging 24 students and 25 students into four groups to do a team work are at the same level of complexity, for a ten-year-old pupil (expected to deal with the operation of division as the second order symbolism) the latter case is conceptually more difficult as it requires the interpretation of the meaning of remainder [55]. Likewise, for a middle school student, the problem of finding on a geoboard the rectangle of the given perimeter with the smallest area is didactically coherent – such rectangle has the one linear unit side length. However, for a high school student such a problem when posed in the Euclidean plane is both numerically and pedagogically incoherent. These examples demonstrate the complexity of interplay that exists among contextual, pedagogical, and numerical coherences of a problem. Furthermore, as will be shown below, in the digital age new methods of resolving mathematical queries emerge, thus adding another dimension to the notion of pedagogical coherence.

3.4. Didactical coherence

The above three coherences can be presented in the form of a Venn diagram (Figure 2). It shows that ideally, a problem posed to students has to belong to the intersection of the three coherences where the problem becomes *didactically coherent*. For example, the problem about recess and soccer, through the lens of the second grader, belongs to region III and it is only the teacher's intervention that placed this problem modification into the didactical coherence region. Attending to the notion of didactical coherence of a problem allows for a greater effect of computational support on problem posing. The Venn diagram of Figure 2, being a cultural tool itself, didactically supports technology-enabled problem posing. The diagram can be seen as a thinking device that scaffolds teachers' ability using "comprehension-fostering and self-regulatory cognitive and metacognitive strategy … [through] question generation" [77, p. 1135]. By learning to use the Venn diagram as a tool that informs problem posing, teachers develop higher order thinking and reasoning skills and gain valuable research-like experience in preparing their own curriculum materials.

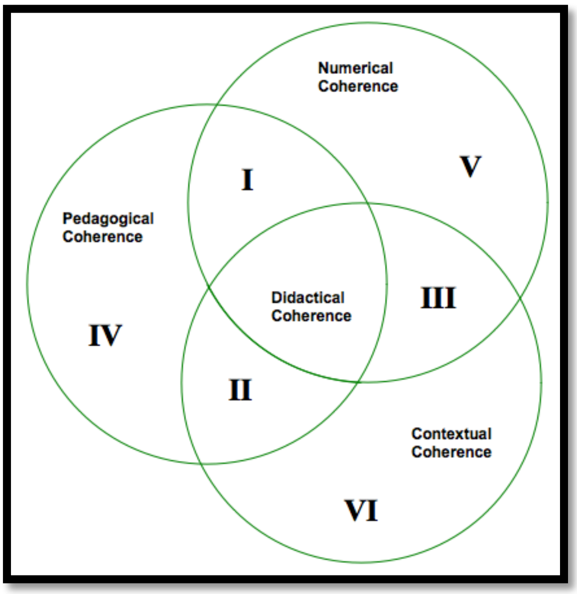

Figure 2. Regions I – VI and Didactical Coherence have empty intersection.

4. Posing TITE problems and collateral learning

By using technology for numeric/symbolic/graphic modeling of pedagogically grade-appropriate mathematical problems, one can pose questions about rather challenging mathematical situations, which, otherwise, are unlikely to be recognized. That is, by furthering teachers' skills in posing questions in the context of technology use, mathematics educators do advance one of the major goals of developing TPCK – "prepare teachers for a classroom environment where technology significantly impacts and changes teaching and learning" [53, p. 510]. Such changes in mathematics classroom pedagogy, in the spirit of Dewey [24], bring about multiple opportunities for collateral learning. This kind of learning is taking place within a context that is much broader than a topic of any given lesson. In this extended context, one is expected and even encouraged to develop links among seemingly disconnected mathematical ideas and concepts; that is, to go far beyond ("fallacious", as Dewey put it) learning "only one particular thing he is studying at the time" [24, p. 49]. Through developing mathematical connections of that kind, one can quite unexpectedly enter the domain of hidden mathematics curriculum [4] within which teachers can be shown "the places where the learner might step into the learning process of mankind" [31, p. ix]. In turn, this allows one to pose technology-immune/technology-enabled (TITE) problems [3], the process of solving of which is both dependent on the use of a computer and resistant to symbolic computations as a workable method of finding an answer. The importance of these kind of problems is two-fold: they support the modern emphasis on using technology for mathematics teaching and learning and take into account the availability of powerful computer programs, like *Wolfram Alpha*, capable of, in response to a natural language input, easily solving traditional problems by performing symbolic computations of different levels of complexity. A TITE problem is pedagogically coherent by design as the method of its solution integrates both argument and computation while balancing the development of mathematical thinking and the use of technology.

As an illustration, consider a historically famous problem of finding the sum of the first 100 natural numbers. According to a legend [28], great Gauss found the answer, 5050, almost immediately after the problem was offered to him (and other students) at the primary school. The teacher of Gauss was expecting the class to be quite busy by adding 100 numbers one by one. Yet, in solving the problem, as the legend goes, Gauss demonstrated what in modern terms is called epistemic fluency [48] – the ability to comprehend and use different ways of knowing and knowledge construction – by recognizing in the sum to be found a multiple of the arithmetic mean of the first and the last addends. In the time of Gauss (19^{th} century) such ability was the only way of circumventing the burden of computation.

In the digital age, posing the query "What is the sum of the first *n* natural numbers?" to *Wolfram Alpha* immediately yields the expression $n(n+1)/2$. This reduction in complexity of solving algebraic problems enables the corresponding epistemic game – "the set of rules and strategies that guide inquiry" [19, p. 25] within a particular representational structure – to be reduced essentially to a simple push of a button. Therefore, a new educational task for problem posing in the digital age is to revisit traditional problems in order to make the use of technology more cognitively demanding despite its automatic problem solving capability. In other words, a new challenge for teachers is to learn posing TITE problems.

5. Two styles of assistance in posing problems

According to Pólya [58, p. 1], "One of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion, and sound principles". In this regard, just as two types (Type I and Type II) of application of technology can be observed in the classroom [46], teachers can offer two styles of assistance to their students [7]. Style I assistance parallels Type I application meaning that teacher's help is limited to providing a student with "right-wrong" evaluations. Likewise, a teacher with partial understanding of what a specific technological tool can do would only offer Style I assistance in the context of Type I application. However, in the context of Type II application of technology, "right-wrong" evaluations and superficial knowledge of technology are likely to be didactically inadequate. For example, Style I assistance can also lead to a phenomenon termed in [36, p. 205] "a localized relentless determination" when a teacher, perhaps unintentionally, limits students' computer work by not appreciating the power and merit of a multiple application environment.

Teachers who are capable of providing Style II assistance are those "who are already more [epistemically] fluent than you are – and who, crucially, are willing to gradually *pull you up* to their level of expertise" [48, p. 114, italics in the original] within a constructivist learning environment. By the same token, Style II assistance may yield both expected (by a teacher) and unexpected (by both parties) learning outcomes. In the digital age, Style II assistance in mathematics requires that a teacher possesses a number of skills and abilities, both technological and mathematical. Whereas knowledge of technology can support its Type II application, knowledge of mathematics can support posing and solving TITE problems. In combination, teachers' (grade appropriate) epistemic fluency in both technology and mathematics constitutes the core of Style II assistance enabling Type II technology applications in the classroom, including posing problems. As discussed in [7], a second grade student participating in a spreadsheet-supported afterschool mathematics/science project was so excited about a possibility of using a computer to do computations, that she formulated a problem of finding a five day temperature range with the answer 95 degrees. While this problem is numerically coherent (e.g., $0^{\circ}, 20^{\circ}, 40^{\circ}, 65^{\circ}, 95^{\circ}$), it is not contextually (and, consequently, not pedagogically) coherent (even when temperature is measured in Fahrenheit). This example shows the importance of the notion of Style II assistance needed to explain to a child the deficiency of the posed problem.

In the context of open-ended problem solving, such help, or, alternatively, assistance can only be of Style II as only this kind of assistance manifests what Pólya [58] referred to as a sound principle. More specifically, in the context of Type II technology application, skills needed for Style II assistance include such elements of TPCK as: (i) recognizing whether a problem is technology immune and if not, knowing how to turn it into a TITE; (ii) knowing how to turn a computer (artifact) into a specific computational learning environment (instrument); (iii) appreciating the notion of instrumental genesis which affects ways an individual uses the instrument [36]; (iv) knowing how to use the instrument to enable solution of a TITE problem and motivate its reasonable extensions thus fostering metacognitive skills through problem posing [77]; (v) having experience with using the results of solving a traditional problem to develop new TITE curriculum materials. The next section will illustrate how Style II assistance can be offered to teachers engaged in posing TITE problems.

6. Style II assistance in problem re-formulating leading to TITE problems

Because pedagogical coherence of a problem not only depends on an expected method of solution but also on context within which it is posed, the re-formulation of a problem possessing some kind of didactic incoherency by changing its parameters in one or other way and employing technology tools can be a problem-solving strategy. Altering context within which a problem has been originally formulated, or even creating of a new context in the case of a context-free problem, while not affecting its formal solvability, can make a problem more appealing in a number of respects and, thus, pedagogically more appropriate for a larger population of students. Also, re-formulation of problems provides multiple opportunities for collateral learning within a hidden mathematics curriculum.

6.1. From numerical to contextual coherence

As an example, consider the following problem used in the pre-digital age in support of afterschool activities of secondary school students with special interest in mathematics [67]:

Problem 4. *Natural numbers are put in groups as follows: (1), (2, 3), (4, 5, 6), (7, 8, 9, 10), (11, 12, 13, 14, 15), ... Find the sum of numbers in the 10th group. Find the sum of numbers in the nth group.*

This (purely numeric) problem is numerically coherent as the set of natural numbers is closed under the operation of addition. That is, the sum of numbers exists regardless of the group's rank. However, the problem does not have any context to be used as first order symbolism in order to facilitate a transition to the second order symbolism where generalization can take place. Just as "the written language of children develops ... [by] shifting from drawings of things to drawings of words" [74, p. 115], the use of formal mathematical symbolism develops by ascribing quantitative meaning to concrete objects (used as frames of reference) that one can manipulate and/or visualize. With this in mind, Problem 4 was included in the course "Creative problem solving" taught by the author for (K-12) teachers to discuss its possible modifications and their place among the regions of the Venn diagram of Figure 2. As is, the problem belongs to region V – it possesses an obvious numerical coherence, has no context, and offers little grade appropriate pedagogy due to the absence of any frame of reference to be used as the first order symbolism. The task for teachers was to reformulate Problem 4 to allow for an exoteric context and engaging pedagogy, thereby, making it accessible to a broader population of secondary school students whose interest in mathematics is in its infancy. To this end, one of the teachers posed

Problem 5. *A group of people is in a room together for some kind of meeting. Each person is expected to become part of a group. The first group will have one person, the second group will have two people, the third group will have three people, and so on. The person in the group one is assigned the number one. The persons in the group two are assigned the numbers two and three. The persons in the group three are assigned the numbers four, five, six, and so on with the remaining groups. Each person in each group is given a piece of candy according to their number. For example, person one gets one piece of candy, person two gets two pieces of candy, person three gets three pieces of candy, and so on. The person handing out the candy wants to put each group's candy in a zip lock bag prior to handing* *it out and needs to know the total pieces of candy each group will get. Help this person to solve the problem.*

A classroom discussion of the reformulated problem then ensued. It was concluded that whereas Problem 5 does have conventional context appealing to students, its pedagogy does not offer any support system in dealing with finding the sum of consecutive natural numbers in a generalized situation (e.g., when such a sums begins with a number different from one). The class decided that Problem 5 belongs to region III of the Venn diagram of Figure 2 meaning that it is numerically and contextually coherent and lacks pedagogical coherence to be considered a didactically coherent problem. In order to facilitate an argument and motivate extended problem posing, a spreadsheet environment for calculating the number of candies in each bag was constructed by the course instructor from where the sequence 1, 5, 15, 34, 65, 111, ... (column A in the spreadsheet of Figure 3) was derived showing the number of candies in a bag (or, in the context of Problem 4, showing that the sum of numbers in the 10th group, located in the region C11:L11, is equal to 505). Below it will be shown how this sequence can be generalized to the *n*-th bag (alternatively, to the *n*-th group in Problem 4) using technology.

	A	C	D	E	F	G	H	I	J	K	L
1	sums										
2	1	1									
3	5	2	3								
4	15	4	5	6							
5	34	7	8	9	10						
6	65	11	12	13	14	15					
7	111	16	17	18	19	20	21				
8	175	22	23	24	25	26	27	28			
9	260	29	30	31	32	33	34	35	36		
10	369	37	38	39	40	41	42	43	44	45	
11	505	46	47	48	49	50	51	52	53	54	55

Figure 3. A spreadsheet showing numerical coherence of Problems 4 and 5.

In particular, modeling data provided by the spreadsheet of Figure 3 allowed for the formulation of

Problem 6. *There are 111 [Figure 3, cell A7] candies in a bag prepared for one of the groups. What can be said about this group?*

This variation of Problem 5 is a TITE problem as its solution integrates argument and computation provided by technology. Problem 6, however, is not obviously numerically coherent. Its numerical coherence is obvious to a problem poser but not to a problem solver. This, however, is the major characteristic of technology-enabled problem posing – whereas problem posers (e.g., teachers) possess knowledge about their problem's solvability, problem solvers (e.g., students) when encountering the problem not only don't know the answer but are unsure whether it even exists.

A problem solver may begin solving Problem 6 in a traditional way by denoting x to be the smallest number of candies a person gets from the bag with 111 candies prepared for n people. Carrying out the summation of n terms of the arithmetic series x, $x + 1$, $x + 2$, … , $x + (n - 1)$ yields the equation $(2x+n-1) / 2 = 111$, which is equivalent to the quadratic equation $n^2 + (2x - 1)n – 222 = 0$. In order for the last equation to have an integer solution, its discriminant $(2x - 1)^2 + 888$ has to be a square number. A computational task is to find the smallest value of x under this condition. Once again, *Wolfram Alpha* can find that $31^2 + 888 = 43^2$. Therefore, $2x -1 = 31$ whence $x = 16$ and $n = (- 31 + 43)/2 = 6$. That is, 111 candies from this bag can be distributed among six people through the following partition: $111 = 16 + 17 + 18 + 19 + 20 + 21$. Note that whereas it is quite appropriate to deal with the expression $(2x - 1)^2 + 888$ using technology, the very development of the quadratic equation was technology immune and thereby, Problem 6 is indeed a TITE problem.

6.2. From modeling data to a general formula using technology

In order to obtain the general formula for the sequence 1, 5, 15, 34, 65, 111, ... , one can enter it into the input box of *Wolfram Alpha*. As a result, the expression $(n^3+n)/2$ is returned as the general answer to Problem 5/Problem 4. (Alternatively, one can use the On-line Encyclopedia of Integer Sequences (OEIS®, https://oeis.org/) which, in addition to the cubic expression, provides multiple, purely mathematical, contexts for the sequence). Such an ease of making generalization through the affordance of the modern day on-line digital tools justifies the need for posing TITE problems in which both mathematical argument and computation (both numeric and symbolic) go hand by hand to allow for the appropriate use of technology. One cannot ignore the existence and, most importantly, availability of these tools because artificially excluding technology from doing mathematics by students does not advance the preparation of the 21st century workforce. To a certain extent, not only Problem 6 but Problem 5 as well may be considered a TITE problem: whereas the generalization phase can be outsourced to *Wolfram Alpha* or OEIS®, computing the number of candies in each bag can be carried out in the paper-and-pencil environment. In addition, the creation of the spreadsheet shown in Figure 3 can become an agency for mathematical activities by teachers. Just as “the teaching [of a child] should be organized in such a way that reading and writing are *necessary* for something” [74, p. 117, italics added], mathematical problem posing and solving as intellectual activities for teachers should be given an applied flavor as much as possible.

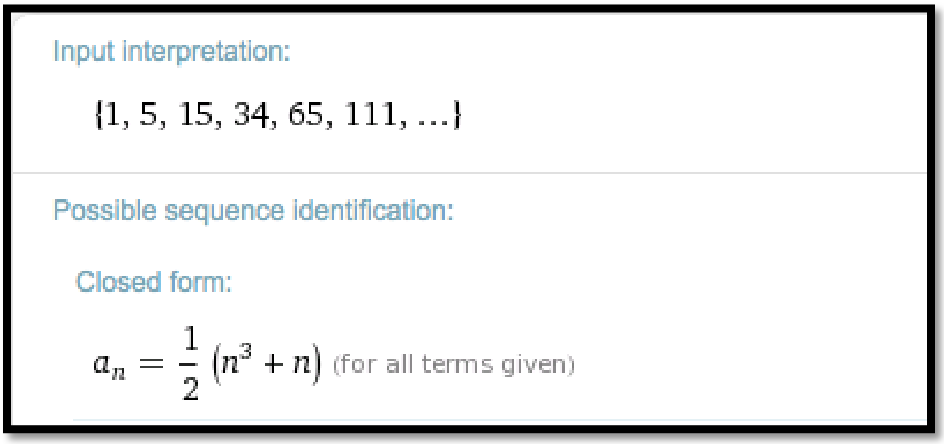

Figure 4. Using *Wolfram Alpha* in generalization.

6.3. Formulating a didactically coherent problem

Whereas bags of candies provide context for dealing with abstract nature of the strings of consecutive natural numbers (the second order symbols), this context does not allow one to manipulate and/or visualize concrete objects involved in order to simplify the summation of numbers. Thus our goal is to design an appropriate problem-solving pedagogy that ultimately would lead to a didactically coherent problem. Such pedagogy can be grounded in recourse to geometry the images of which can be used as the first order symbols. Towards this end, another teacher formulated

Problem 7. *John is making towers from blocks according to the pattern shown in Figure 5. How many blocks does he need to build the 10th tower in this pattern? How many blocks does he need to build the n*th *tower in this pattern?*

It should be noted that the formulation of Problem 7 was due to the teacher’s experience with similar problems such as constructing towers out of matchsticks (e.g., [4, 32, 52]). In addition, familiarity with the latter type of problems motivated the teacher to offer an extension of Problem 7 when the towers are constructed out of matchsticks to allow for a different type of counting techniques to be considered.

Such counting of the matchsticks and its generalization to an arbitrary number of towers provide another interesting example of a TITE problem.

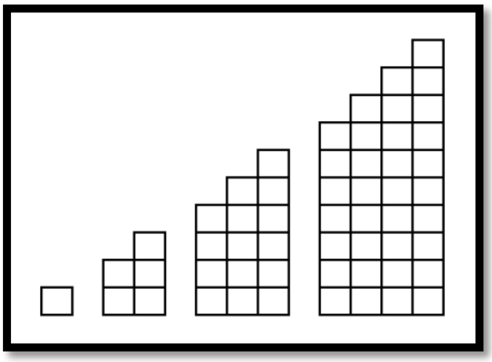

Figure 5. John's towers built out of blocks.

In a mean time, by analyzing step-towers of Figure 5, one can *see* that each such tower comprises components the heights of which are consecutive natural numbers. Now the sum of numbers in each group (Problem 4) or the number of candies in each bag (Problem 5) coincide with the number of blocks used to build the corresponding tower. One strategy to find the number of blocks in a tower of this structure is to augment it to have a rectangle which area is easy to find (Figure 6). While a basic tower represents geometrically consecutive natural numbers the smallest of which (beginning from the second tower) is different from one, the augmentation always represents consecutive natural numbers the smallest of which is equal to one. Thus, the number of auxiliary blocks can be found without much difficulty. In each tower, its rank in the set of towers is the width. The height can be found in a number of ways. One way is to associate the height with the largest number in each group of the original problem. The sequence of numbers is 1, 3, 6, 10, 15, ... Once again, using *Wolfram Alpha* yields a familiar expression, $n(n+1)/2$. Consequently, the number of blocks in the n-th tower is equal to $n \cdot n(n+1)/2 = n^2(n+1)/2$. At the same time, the number of blocks augmenting the n-th tower to a rectangular shape equals to the height of the $(n-1)$-th tower, that is, $n(n-1)/2$. Therefore, the number of blocks in the n-th tower gives the sum of numbers in the n-th group as follows:

$$n^2(n+1)/2 - n(n-1)/2 = n(n^2+1)/2.$$

One can say that Problem 7 is a TITE problem: it could not be solved through what Guin & Trouche [36] called "an *automatic transport phenomenon* ... [when the outcome of problem solving process depends on whether] one can feed all the problem's data into the machine" (p. 205, italics in the original). Yet technology was used in finding the general form of the sequences 1, 5, 15, 34, 65, 111, ... and 1, 3, 6, 10, 15, ... However, the intricacy of human-computer interaction is that a skill of feeding a number sequence into *Wolfram Alpha* should not be taken for granted. The last remark will be illustrated in section 8 below.

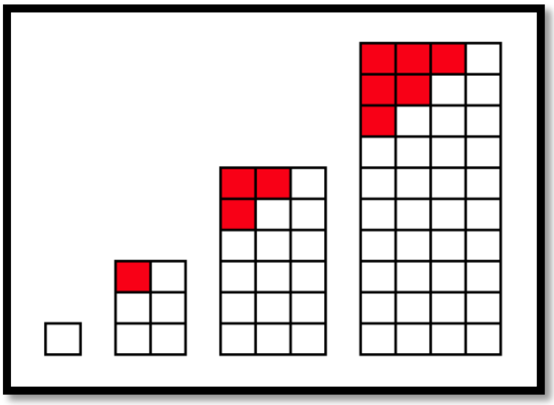

Figure 6. Augmenting towers to facilitate counting of blocks.

7. Hidden mathematics curriculum and problem posing

The notion of hidden mathematics curriculum of teacher education (Abramovich, 2009) stems from an observation that many seemingly unrelated mathematical concepts and problems typically associated with the K-12 mathematics curriculum turn out to be truly connected through a common conceptual structure which is hidden from inexperienced learners because of its deep-rooted complexity. Moreover, even mathematical activities traditionally treated by teachers as routine ones, when examined beyond the boundaries of a studied topic, can be used as windows on big ideas that are not obviously portrayed by the curriculum. Rather, such ideas have to be naturally uncovered by a 'more knowledgeable other' and then explicitly introduced to teachers.

Utilizing the hidden mathematics curriculum framework in the broad context of mathematics teacher education provides multiple opportunities for collateral learning in the spirit of Dewey [24] by showing teachers the places where mathematics can be explored in more depth than the traditional curriculum recommends. In particular, problem posing has great potential to create entries into so hidden mathematics curriculum by encouraging teachers to plausibly reflect on problems they themselves have already posed and/or solved. Such encouragement from those involved in the teacher preparation requires significant (grade appropriate) epistemic fluency [48] as the core element of Style II assistance.

With this in mind, Problem 5 can be extended to the bags of different kind when the number of candies in a bag and their distribution follows the old pattern, yet the number of group members may vary. For example, there are three ways to distribute 15 candies: 15 = 4 + 5 + 6 (Problem 5), 15 = 7 + 8, and 15 = 1 + 2 + 3 + 4 + 5; yet the larger bag does not provide the increase of distributions: e.g., for the fifth bag we have only two ways to distribute candies, namely, $65 = 11 + 12 + 13 + 14 + 15$ and $65 = 2 + 3 + 4 + ... + 11$. Hidden in this simple (though not necessarily obvious) observation is the concept of trapezoidal numbers introduced in [59]. These numbers can be partitioned into a sum of consecutive natural numbers and, as the above two examples show, possibly in more than one way (not counting a trivial partition into itself, like 15 = 15). In particular, the total number of partitions of the number *N* into a sum of consecutive natural numbers is equal to the number of its odd divisors. For more information on this topic and its connection to technology-enhanced teacher education including computational derivation of the partitions see [1].

In that way, using the concept of trapezoidal numbers as an element of hidden mathematics curriculum prompts posing the following two problems.

Problem 8. *In the context of Problem 5, find the smallest bag in which candies can be given to other groups of people so that the sequence of numbers representing candies given to each person form consecutive natural numbers. Can this be done in more than one way? Why or why not? Can you find another such bag?*

Problem 9. *In the context of Problem 7, find the smallest tower that can be re-built in such a way that the heights of its components are consecutive natural numbers. Can the tower found be re-built in more than one way? Why or why not? Can any of the John's towers be re-built to have the heights consecutive natural numbers?*

Note that the above two problems are just examples of how knowledge of hidden mathematics curriculum, that is, knowing "the place where the learner might step into the learning process of mankind" [31, p. ix] enhanced by the appropriate computational environments can be used by teachers for posing quite challenging problems to their students. Teachers need to have experience with questions that are not easy to answer. This is one of the main characteristic features of mathematics when a slight change in the condition of a problem makes it difficult to solve even if it remains numerically coherent. As for the last two problems, they are not obviously numerically coherent while their contextual and pedagogical coherences depend on the level of students to whom the problems are offered.

8. Learning to pose problems for software to solve

Sometimes, a computer program (e.g., *Wolfram Alpha*) can offer several options for a student to select the correct answer. Consequently, one's ability to navigate through these options might require knowledge of mathematics beyond the grade level involved. For example, entering the equation

$$LW + (L - W) = 183$$

a mathematical model for the problem from Babylonian mathematics mentioned in section 2, yields a reference to hyperbola (as a locus of the equation) and to 16 integer solutions, both negative and positive. This mathematical variety of interpretations of the equation might be a confusing factor for an inexperienced user of *Wolfram Alpha*. Therefore, the diversity of thinking encouraged by the current standards for the teaching of mathematics might be ill advised by a computer. This raises the need for teachers to develop skills in posing a problem for a computer to solve enabling multiple interpretations of an answer to be reduced to minimum. Recall the case of the second grader who saw the information “9 did not play soccer” as irrelevant in the context of recess. So, the issue of didactical coherence of a problem can be extended to TITE problems that integrate argument and computation.

For instance, entering only four terms – 1, 5, 15, 34 – of the sequence that describes the number of candies in the bags (Problem 5) into the dialogue box of *Wolfram Alpha* does not produce a general formula of this sequence. The program needs two more terms to understand the query. Another example deals with mathematics terminology. Although *Wolfram Alpha* defines a counting number as a natural number, the query “What is the sum of the first *n* counting numbers?” does not produce the expression $n(n + 1)/2$ like it was in the case when the word *natural* was used instead of the word *counting*. Such simple cases demonstrate the importance of developing skills in posing questions/problems to *Wolfram Alpha*. This kind of intricacy of human-computer interaction calls for a strong (grade appropriate) knowledge of mathematics by teachers.

Consider another example. The spreadsheet of Figure 3 has generated in column E the sequence of numbers 6, 9, 13, 18, 24, 31, ... One can pose the problem of finding the sum of the first 100 terms of this sequence and enter the expression

$$6+9+13+18+24+31+...$$

into the input box of *Wolfram Alpha*. The program responds with the m^{th} partial sum formula (Figure 7) which allows for finding the sum of the first 100 terms of the sequence that is equal to 177150. Similar problems can be formulated by using number sequences generated in other columns of the spreadsheet of Figure 3. If, for some reason, *Wolfram Alpha* is unable to determine a partial sum formula, rather that inputting the expression $6+9+13+18+24+31+...$, one can enter the sequence $6, 9, 13, 18, 24, 31, ...$ for which the general formula $(n^3+3n+8)/2$ results. In turn, by inputting the quest “Find the sum of $(i^3+3i+8)/2$ from i = 1 to i = n” yields the partial sum formula. A teacher’s ability to alter the formulation of problems for software to solve forms the confluence of knowledge of mathematical content and the appropriate use of technology.

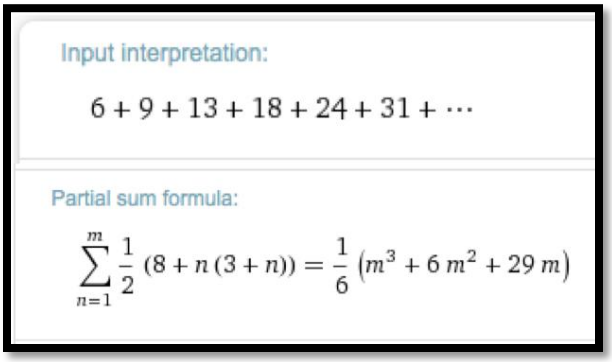

Figure 7. Finding a partial sum formula using *Wolfram Alpha*.

9. Analyzing a formula as a TITE problem

One can use *Maple* in posing a task of proving by the method of mathematical induction a certain proposition depending on an integer variable. In the previous section, the expression $n(n+1)/2$ was shown representing the sum of the first *n* natural numbers;

that is, the formula $1+2+3+...+n=n(n+1)/2$ holds true for any $n\in\boldsymbol{N}$. In particular, this implies that the product $n(n+1)$ is divisible by two for any $n\in\boldsymbol{N}$. Therefore, one can pose

Problem 10: *Prove by the method of mathematical induction that* $n(n+1)$ is *divisible by two for any* $n\in\boldsymbol{N}$.

According to Pólya [57], the main idea of this method is to "test the transition from n to $n+1$" (p. 111). In the context of *Maple*, assuming that $P(n)=n(n+1)$ is divisible by two and showing that the same is correct for the difference $P(n+1) - P(n)$ implies that $P(n+1)$ is divisible by two. *Maple* yields the relation $P(n+1) - P(n) = 2n+2$ the right-hand side of which, regardless of n, is obviously a multiple of two.

Similarly, in the previous section the expression $(n^3+6n^2+29n)/6$ was shown representing an integer number sequence $6, 9, 13, 18, 24, 31, ...$. Although the expression looks like a fraction, it produces integers only and, thereby it has to be divisible by six for any $n\in\boldsymbol{N}$. This observation can lead to posing

Problem 11: *Prove by the method of mathematical induction that* n^3+6n^2+29n is *divisible by six for any* $n\in\boldsymbol{N}$.

This time, testing the transition from n to $n+1$ cannot be completed by *Maple* in one induction step. Assuming that $P(n)=n^3+6n^2+29n$ is divisible by six yields, as shown in Figure 9, that

$$P(n+1) - P(n) = 3n^2+15n+36$$

a trinomial not obviously divisible by six. Therefore, *Maple* has to be asked again to prove this divisibility by mathematical induction. To this end, assuming that the expression $Q(n)=3n^2+15n+36$ is divisible by six, one has to show that the difference $Q(n+1) - Q(n)$ is divisible by six as well. *Maple* shows the difference $P(n+1) - P(n) = 6n+18$. Recognizing that $6n+18$ is an obvious multiple of six completes the proof of divisibility by 6 of the trinomial n^3+6n^2+29n.

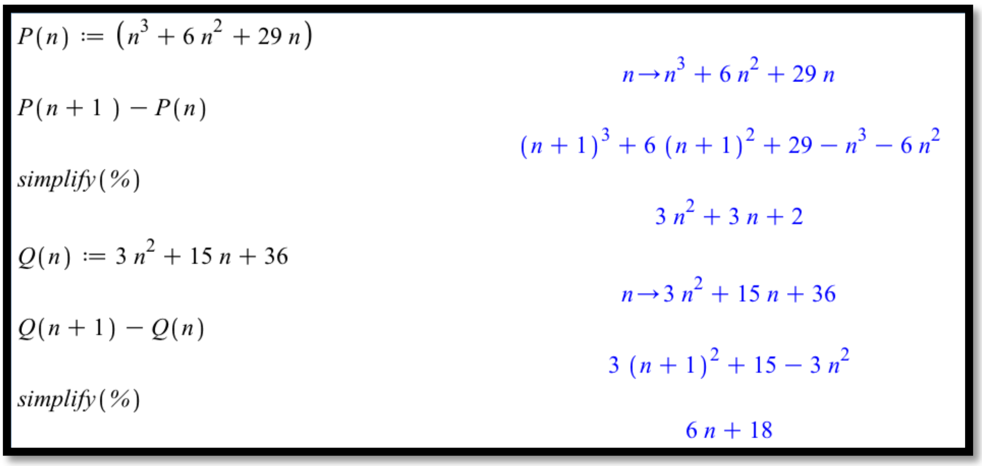

Figure 8. Testing divisibility of n^3+6n^2+29n by 6 in two steps using *Maple*.

Likewise, by demonstrating in the context of *Wolfram Alpha* that the n-th partial sum of the series $10+14+19+25+32+40+...$ is equal to the cubic trinomial $(n^3+9n^2+50n)/6$, one can prove using the *Maple*-based method of mathematical induction that $(n^3+9n^2+50n)/6$ is divisible by six for any $n\in\boldsymbol{N}$. Note that such use of *Maple* demonstrates how one can deal with appropriately posed TITE problems by outsourcing symbolic computations to the software. This novel way of using technology should not be taken to mean that human-computer interaction is always straightforward even within an apparently identical mathematical context. Indeed, unlike the case of Problem 10 where proof required a single induction step, an apparently similar case of Problem 11 required two induction steps. Such intricacy of the *Maple*-based mathematical induction proof within a seemingly same class of TITE problems indicate the importance of the notion of Style II assistance in the problem-posing and problem-solving contexts of the digital age.

Although algebraic transformations required for demonstrating the transition from n to $n + 1$ in the above examples are not really challenging and, perhaps, should belong to the (secondary mathematics) teachers' mathematical machinery tool kit, the following perspectives on the use of technology in mathematics teaching appear to be appropriate noting in the conclusion to this section. In the digital era traditional paper-and-pencil techniques can give way to new techniques made possible by the appropriate use of technology [27]. Such a use can be seen as a part of one's instrumental genesis through which new ideas about the appropriation of technology to support a particular content develop [36]. As Langtangen & Tveito [44] put it, "Much of the current focus on algebraically challenging, lengthy, error-prone paper and pencil work can be significantly reduced. The reason for such an evolution is that the computer is simply much better than humans on any theoretically phrased well-defined repetitive operation" (pp. 811-812).

10. Conclusion

The material of this paper was motivated by the author's work with (K-12) teachers in a number of technology-enhanced mathematics education courses. One specific focus of these courses was to foster problem-posing skills enabled by the technological advancements of the digital age. Analyzing problems posed by the teachers and their use of technology that was available (and, in some cases, specifically designed) in support of the courses has led to the development of two distinct yet interrelated notions: the didactical coherence of a problem and the TITE (technology-immune/technology-enabled) problem.

Posing problems, in general, can be interpreted as a venture of asking questions (e.g., [35]). One such question that was emphasized in this paper is quite typical for mathematics: "How many?" As shown throughout the paper, through this kind of questions, quantitative information about a certain situation or phenomenon was sought. Depending on a grade level, such information can be obtained either through making an organized list or using theory. That is, for problem solvers (e.g., students) no technology was expected to support answering the "how many" questions. On the contrary, problem posers (e.g., teachers) described in this paper did see all the answers to their questions within a computational medium, which was specifically designed to support their problem-posing practice. The ease of computing made it possible to alter numeric data to enable two components of didactical coherence of a problem – numerical and pedagogical coherences. The third component, contextual coherence, was independent of the ease of computing requiring teachers' cultural sensitivity and social competence. That is, contextual coherence was technology immune.

Posing TITE problems further required a certain level of pedagogical, mathematical, and technological sophistication on the part of teachers to enable the development of curriculum materials appropriate for the digital age. In other words, the appreciation of alternative ways of mathematics curriculum design required the teachers' ownership of the corresponding technological pedagogical content knowledge. The latter concept provides solid theoretical framework for emerging research on the formulation of TITE problems.

The concluding remarks are aimed at analyzing problems discussed in this paper as an art of asking mathematical questions informed by a number of theoretical considerations. In this analysis, one can make a distinction between questions seeking information or explanation as such and questions requesting specific types of explanation [37]. Here, one can distinguish between questions that don't presuppose reflection and questions motivating reflective inquiry into a quantitative result. The latter type of questions, often stemming from the thorough analysis of computer-generated data, may be considered more intelligent. Often, this intelligence requires just a slight modification of questions that request explanation of the information obtained. Problems asking for explanation are reflections on problems requesting information. In the context of mathematics, problems that are seeking an explanation of a certain phenomenon may indeed be more challenging than those through which a certain phenomenon formulated in quantitative terms can be revealed. In analyzing a problem, one can be asked whether it is didactically coherent and if not, how the alleged deficiency in its formulation can be rectified. For example, one such problem, as a reflection on Problem 3 (section 3.3) can seek the following specific explanation: Why couldn't Carol spend exactly 50 cents by buying some 9-cent stamps and as many as 5-cent and 1-cent stamps? As was mentioned above, the information presented in the last question does not lead to an answer as the resulting equation, $9x + 6y = 50$, does not have whole number solutions. In other words, the problem is not numerically coherent. Technology has great potential to help teachers move from asking questions requesting information as such to raising queries that are reflections on those questions.

Problems of a more open-ended nature like Problems 8, 9 (section 7) do motivate and even encourage reflection by asking whether something could be done in more than one way. Furthermore, the syntactic query of the form "Why or why not" demands explanation of what was discovered through reflection. This explanation is not trivial and it requires knowledge of hidden mathematics curriculum and corresponding computational tools where collateral learning of trapezoidal numbers and their properties can take place. A complicated nature of explanation needed to address a mathematical query should not discourage teachers from posing new questions. On the contrary, this experience prepares them to handle the inherent uncertainty of a mathematical discourse where a simple question often requires an answer far beyond the grade level involved, even in the digital age. Such preparation can significantly enrich the PCK of teachers and their skills in the Type II technology application, the joint use of which is what comprises their TPCK.

Problems on a *Maple*-supported proof are requesting explanation of already obtained information. For example, in the context of Problem 11, demonstrating that the assumption about the trinomial $n^3 + 6n^2 + 29n$ being divisible by six implies the divisibility by six of

$$(n+1)^3 + 6(n+1)^2 + 29(n+1)$$

is a way of explaining, in the language of formal mathematics, already received information from a computational tool that the former trinomial is the general term for a certain integer sequence. The art of asking questions that seek explanation of a specific mathematical phenomenon revealed by technology should be especially emphasized in the context of fostering creative thinking in students. It is through this art that research on the problem-posing component of the concept of TPCK can be carried out under a new angle leaning toward alternative ways of mathematics curriculum design in the digital age.

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