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REVISITING MATHEMATICAL ACTIVITIES FOR SECONDARY TEACHERS THROUGH THE LENSES OF MODERN DIGITAL TOOLS

Sergei Abramovich

School of Education and Professional Studies
State University of New York at Potsdam, USA
abramovs@potsdam.edu

Michael L. Connell

College of Public Service
University of Houston-Downtown, USA
connellm@uhd.edu

Abstract. The content of this paper stems from an earlier inquiry into the use of computers in secondary mathematics teacher education. The advent of the modern day digital tools (such as *Maple* and *Wolfram Alpha*) capable of sophisticated symbolic computations calls for the revision of technology uses set forth in the early years of teaching mathematics through problem solving [26]. Nowadays, technology not only facilitates problem solving to the extent of making it just "easy", but provides learning opportunities for deeper inquiries into seemingly sealed for non-professional mathematical investigations that require formal reasoning. The activities described in the article are connected to recent standards for teaching mathematics and recommendations for the preparation of teacher candidates published in the United States and elsewhere in the world. Through the suggested activities one can appreciate the integration of mathematical understanding, conceptual knowledge, procedural skills and technological competence.

Key words: *Maple*, *Wolfram Alpha*, spreadsheet, triangular numbers, proof, TITE problem, TPCK concept, mathematics teacher education.

ZDM Subject Classification: I70, N70, R20

1. Introduction

The Council for the Accreditation of Educator Preparation [14], a group commonly known in the United States as CAEP, recommended that teacher educators strive to "model best practices in digital learning and technology applications that EPP [education preparation provider] expects candidates to acquire" (p. 30). At the Federal level, those working for an EPP have been advised that "computational technology can be a powerful driving force for innovation in education … advancing rapidly to the point that it can soon play a transformational role in education" [31, p. xi]. Whereas educational innovations, coupled with the modern students' characterization as digital natives [30], can transform classroom pedagogy in many significant ways, some studies (e.g., [15], [20]) suggested that being a 'digital native' does not mean that one is prepared to appropriately use technology in the context of academic work without competent guidance of the teacher. One of the implications of those studies in the context of mathematics teacher education is the need for new teaching ideas and instructional materials that support recommendations for teacher preparation emphasizing the importance of "generalizing, finding common structures in theorems and proofs, … and forming connections between seemingly unrelated concepts" [12, p. 56]. By reflecting on an earlier research concerning the use of technology in secondary mathematics teacher education through the lenses of newer digital tools (*Wolfram Alpha* and *Maple*, in addition to the modern spreadsheet), recent standards for teaching mathematics [11], [16], [24], [25], [27], [32], [33] and recommendations for teacher preparation [8], [9], [12], [14], [19] published in the United States and elsewhere in the world, this paper offers several teaching ideas of technology integration that can contribute to the advancement and dissemination among EPPs of the best practices of learning in the digital era.

2. Background information and the goals of the paper

In the early 1990s, when working on a paper about the use of technology for teaching topics in number theory [6], the first author came across [10] several sequences of numbers, among them

$$21, 2211, 222111, 22221111, 2222211111, \ldots \qquad (1)$$

and

$$55, 5050, 500500, 50005000, 5000050000, \ldots \qquad (2)$$

which were referred to as the sequences of triangular numbers – partial sums of consecutive natural numbers starting from one. For example, $1 + 2 + \ldots + 6 = 21$ and $1 + 2 + \ldots + 10 = 55$ – the first terms of (1) and (2), respectively. Yet, already representing their second terms, 2211 and 5050, as partial sums of consecutive natural numbers requires some conceptual understanding of triangular numbers. Such understanding begins with the appropriate contextualization of a concept to be introduced. In the modern practice of mathematics education, "the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved" [11, p. 6, italics in the original] is considered an important element of students' procedural fluency, mathematical competence, and conceptual understanding. With this in mind, note that creating such sums can be put in context in a variety of ways; e.g., by counting handshakes: two people – one handshake, three people – three handshakes (1 + 2), four people – six handshakes (1 + 2 + 3), and so on. So, the number of handshakes among seven people is 21, among eleven people – 55. In general, the sum 1 + 2 + 3 + … + n represents the number of handshakes among n + 1 people. Alternatively, the number of handshakes among n + 1 people can be represented by the fraction $n(n + 1)/2$ (e.g., counting the handshakes twice by using a tree diagram when each of the n + 1 stems supports n branches) from where the equality $1 + 2 + 3 + \ldots + n = n(n + 1)/2$ results. The case of five people (n = 4) handshaking is shown in Figure 1.

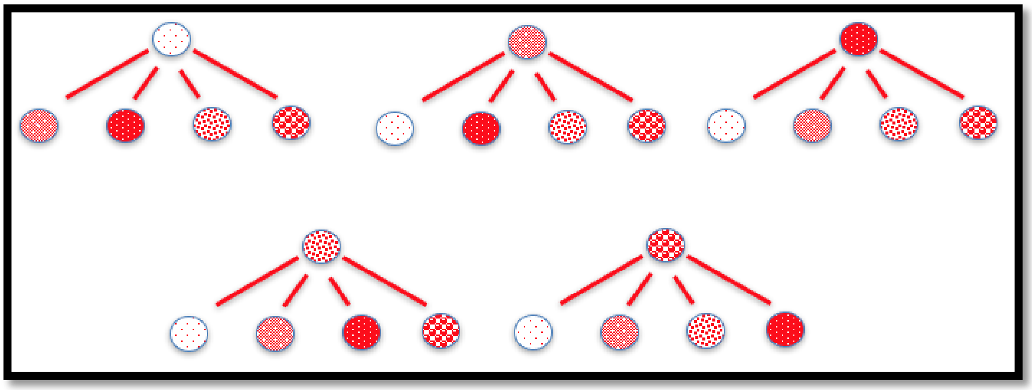

Figure 1. Counting handshakes twice among five people.

This, in turn, motivates the question: How can one show that the terms of sequences (1) and (2) are of the form $n(n + 1)/2$, that is, are half the product of two consecutive natural numbers? For example, factoring 21 yields $3 \cdot 7 = 6 \cdot 7/2$. Solving the quadratic equation $n(n + 1)/2 = 21$ yields $n = \frac{-1+\sqrt{1+8\cdot 21}}{2} = \frac{-1+13}{2} = 6$ implying that because $1+8\cdot 21=169=13^2$ and $(-1 + 13)/2 = 6$, the number 21 is the triangular number of rank six. Likewise, solving the equation $n(n + 1)/2 = 55$ yields $n = \frac{-1+\sqrt{1+8\cdot 55}}{2} = \frac{-1+21}{2} = 10$, implying that the number 55 is the triangular number of rank ten. In order to verify that this property (the square root of eight times the tested number increased by one being an odd integer – known as the square test) holds true for other terms of sequences (1) and (2), a spreadsheet had been suggested [6]. This verification was to introduce a spreadsheet as a powerful computational tool that can motivate prospective secondary mathematics teachers to use technology in the classroom.

The goal of this paper is to revisit the use of technology in exploring sequences like (1) and (2), and to demonstrate how the noted property of these sequences (i.e., being triangular numbers) can be a source of new problems called TITE problems [1], [2], [5], [13]. Such problems are technology immune (TI), for they may not be solved automatically by software at the push of a button. At the same time, they are technology enabled (TE), as their solution and its demonstration can be significantly enhanced by the use of technology. The development of TITE problems can address "a growing need for new instructional materials … that are aligned with higher standards and provide much richer learning experience and more vibrant sources of information" [31, pp. 80-81]. Indeed, the didactic duality of TITE problems allows for the development of the intellectual vigor enhancing one's mathematical understanding and technological competence when learning not only to solve but also to pose problems makes it possible to connect procedural and conceptual knowledge [3], [4]. When solving a problem through a pure argument in the digital era, one decides which part of the argument can and may be computationally supported. It is this decision that provides one with a problem-posing experience, something that is aligned with higher standards of learning mathematics in comparison with the traditional curriculum. Moreover, the TITE problem research and development can address a call for "research on new kinds of assessment and new ways to develop assessments" [31, p. 91]. In the context of TITE problem solving and posing, this paper will demonstrate how one can use the modern-day technology tools not only to find many other sequences of that kind, but to uncover an algorithm of generating such sequences through conceptual, mathematically informed understanding of the algorithm.

3. TITE problems and the Technological Pedagogical Content Knowledge (TPCK) framework

One should not assume that other triangular numbers can serve as seed values allowing for the demonstration of the same phenomenon that sequences (1) and (2) reveal. For example, 28 is a triangular number but 2288 is not. Likewise, 66 is a triangular number but 6060 is not. So, an inquiry into such surprising properties of certain classic sequences of natural numbers requires conceptual understanding of how their terms develop and which triangular numbers may serve as seed values for such sequences. Do the seed values somehow depend on the rank of a triangular number? What is special about such ranks? How can one develop other sequences of triangular numbers with similar properties? Can other sequences of numbers with repeating digits be found to possess specific properties? How can explorations with triangular numbers inform the development of such sequences? These and like questions are worth exploring in technology-enhanced contexts of problem solving and problem posing.

Explorations described in this paper can be used in technology-enhanced courses for secondary mathematics teacher candidates emphasizing a possibility to connect procedural and conceptual knowledge through posing and solving problems in the technological paradigm. Conceptualization of an algorithm which is an element of procedural knowledge in mathematics is needed for enabling an algorithm to work with computational technology. For example, both formulas $x_n = x_{n-1} + n, x_1 = 1$ and $x_n = n(n+1)/2$, recursive and closed, respectively, generate the sequence of consecutive triangular numbers starting from one and can be used effectively to generate the numbers in the context of a spreadsheet. But what are the rules that generate sequences (1) and (2) so that a computer can understand those rules? Such a question brings about a TITE problem. It has a TI component for it requires to create an algorithm through which sequences (1) and (2) can be generated. The problem has also a TE component for the algorithm has to be verified computationally. A computational algorithm is a rule which cannot be created without understanding its genesis. As stated in [11], "mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from" (p. 4, italics in the original). This statement is echoed in [12] for those developing technology-enhanced courses for prospective mathematics teachers, "technology used in superficial way without connection to mathematical reasoning, can take up precious course time without advancing learning" (p. 57). Put another way, a TITE problem does connect mathematical understanding and technological competence. Teacher candidates' ability to formulate a TITE problem can be seen as an important skill that belongs to the TPCK concept [22], [28], allowing the candidates to "advance from novice to expert thinking about designing instruction with technology" [7, p. 162].

4. Two ways of representing numbers with repeating digits

Note that any number of the form $\underbrace{kk\ldots k}_{n}$ where $1 \leq k \leq 9$ is a k-multiple of the number $\underbrace{11\ldots 1}_{n}$, which can be represented in two ways: as the sum $10^{n-1} + 10^{n-2} + \cdots + 1$, and as the fraction $\frac{10^n - 1}{9}$. For example, 111 = 100 + 10 + 1 and $111 = \frac{999}{9} = \frac{10^3 - 1}{9}$. Consequently, the identity

$$10^{n-1} + 10^{n-2} + \cdots + 1 = \frac{10^n - 1}{9} \tag{3}$$

can be formulated and be used as a simple context for learning to do proofs.

While the validity of (3) was demonstrated above for $n = 3$, its real meaning and purpose is to provide an identity of two representations that holds true for all natural numbers n. This general demonstration allows one to introduce two ways of proving identities involving sums: through actual summation and by the method of mathematical induction. By using the formula for the sum of geometric series $1 + 10 + 10^2 + \cdots + 10^{n-1} = \frac{10^{n-1} \cdot 10 - 1}{10 - 1} = \frac{10^n - 1}{9}$, identity (3) immediately results. Assuming that (3) is true, mathematical induction proof consists in the demonstration of the inductive transfer, referred to in [29] as transition from n to $n + 1$, as follows

$$10^n + (10^{n-1} + 10^{n-2} + \cdots + 1) = 10^n + \frac{10^n - 1}{9} = \frac{9 \cdot 10^n + 10^n - 1}{9} = \frac{10^{n+1} - 1}{9},$$

showing that (3) remains true when n is replaced by $n + 1$. This allows one to conclude that because identity (3) is true for $n = 1$, it is true when 1 is replaced by 2, when 2 is replaced by 3, and so on; that is, in general, it is true for any value of n.

While in the case of formula (3) both approaches to proof do not require the use of technology, in the case of more complicated identities their proof may require rather involved symbolic computations, something that can (and perhaps should) be outsourced to a computer. In such cases, proving algebraic identities or other statements depending on an integer variable can be considered in the context of a TITE problem solving and posing. Examples of such tasks will be discussed below.

5. How many numbers of the form $\underbrace{kk\ldots k}_{n}$ are triangular numbers?

Very few. An answer to this question has been known for more than a century. Youngman [34] asked for a proof that the number 666 is the largest triangular number comprised of the same digits and two proofs were presented there in response. More specifically, using certain concepts of number theory beyond the secondary level, the response was that besides 1, 3, and 6 (the case $n = 1$), there are only three more numbers – 55, 66, and 666 – among all integers with at most 30 digits. In the context of the present paper, one can consider exploring this situation as a TITE problem. Its TI part sets the stage for a TE part. To begin the former part, note that in order for the number $\underbrace{kk\ldots k}_{n} = \frac{k(10^n - 1)}{9}$ to be a triangular number, the equation

$$\frac{k(10^n-1)}{9}=\frac{m(m+1)}{2} \tag{4}$$

should have a solution in integers k, m, and n. If such a solution exists, the value of m satisfying equation (4) is the rank of a triangular number sought (i.e., the numbers with the digit k repeated n times).

Obviously, the range for a digit k is [1, 9]. When n = 1 we have three (one-digit) triangular numbers mentioned above. Let the range for n be [2, 9]. To find the corresponding range for m, given the ranges for k and n, one can use equation (4) with the largest values of k = 9 and n = 9. It follows from (4) that $m^2+m-2(10^9-1)=0$ whence $m=\frac{-1+\sqrt{1+8(10^9-1)}}{2}<50{,}000$. The above information, obtained within the TI stage of finding triangular numbers with the same digit repeating over and over, allows one to move to the TE stage and to construct a spreadsheet (Figure 2) which confirms that among the first million integers, the only triangular numbers are 55, 66, and 666. By changing the left end point of the segment that shows the range for m, the spreadsheet investigation can be extended beyond the first million integers. This concludes solving the problem of finding triangular numbers which consist of the same digits.

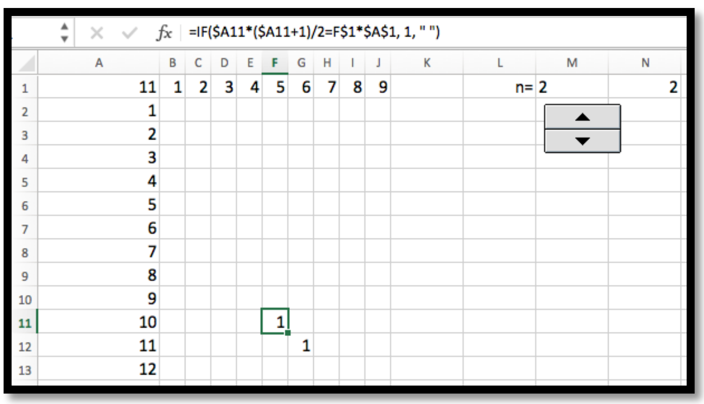

Figure 2. The spreadsheet locates 55 and 66 as triangular numbers with the same digit.

6. Discovering sequence (1) through collateral learning

While there are very few triangular numbers of the form $\underbrace{kk\ldots k}_{n}$, one can consider such integers with repeated digits as the ranks of triangular numbers. The question to explore as a TITE problem is: What is special about the triangular numbers with the ranks $\underbrace{kk\ldots k}_{n}$? To answer this question, note that due to the equality $\underbrace{11\ldots 1}_{n}=\frac{10^n-1}{9}$, triangular numbers with the ranks $\underbrace{kk\ldots k}_{n}$ can be written as follows

$$\frac{k(10^n-1)}{2\cdot 9}\cdot\left(\frac{k(10^n-1)}{9}+1\right)=\frac{k(10^n-1)(k(10^n-1)+9)}{162}\,. \tag{5}$$

One can check to see that when $n = 1$ and $k = 6$, the right-hand side of (5) yields 21 – the triangular number of rank six. The next step may be to create sequences of triangular numbers with the ranks $\underbrace{kk\ldots k}_{n}$ for different values of k by using *Wolfram Alpha* – a computational knowledge engine available free on-line and capable of both numeric and symbolic computations.

To this end, the following simple command has to be entered into the input box of *Wolfram Alpha*:

Table[k(10^n-1)(k(10^n-1)+9)/162, {k, 10}, {n, 10}].

The result of computations is shown in Figure 3. In particular, when $k = 6$ sequence (1) results. That is, by exploring triangular numbers the ranks of which consist of the same digits, sequence (1) resulted in a collateral learning mode. It is through collateral learning that the fact of the entire sequence (1) comprised of triangular number of the ranks $\underbrace{66\ldots6}_{n}$ for all natural values of n was discovered. In other words, the appropriate use of technology the preparation for which occurs within a TI stage of solving a TITE problem creates conditions for the emergence of what Dewey [17] called collateral learning, emphasizing the significance of this kind of learning through the following tenet: "Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time" (p. 49). In the context of this paper, by exploring triangular numbers the ranks of which are comprised of repeated digits, it was discussed that the numbers presented by sequence (1) are triangular numbers the ranks of which consist of the digit 6 only repeated as many times as either of its digits. For example, 21 and 2211 are triangular numbers of the ranks 6 and 66, respectively.

In that way, a closed formula for sequence (1) can be written in the form

$$\frac{6(10^n-1)(6\cdot10^n+9)}{162}=\frac{(10^n-1)(2\cdot10^n+1)}{9}.$$

Setting $x_n=\frac{(10^n-1)(2\cdot10^n+1)}{9}$ yields $x_n - x_{n-1} = 10^n(22\cdot10^n-1)$ and, therefore, the recursive formula for sequence (1) has the form

$$x_n = x_{n-1} + 10^n(22\cdot10^n-1), x_1 = 21.$$

Note that proving that sequence (1) consists of triangular numbers only was not straightforward. It came as a result of exploring numbers generated by the sequences of units; that is, the genesis of the problems explored in this paper can be found in this sequence of numbers showing the pivotal role of the unity in the development of mathematical concepts. This fact is important for prospective secondary teachers to appreciate because the units serve as building blocks of mathematics. Below, it will be shown how the discovered characteristic of the terms of sequence (1) can be revealed through its direct investigation.

Furthermore, when $k = 3$, another interesting sequence of triangular numbers can be derived from the table (Figure 3) generated by *Wolfram Alpha*:

$$6, 561, 55611, 5556111, \ldots, \underbrace{55\ldots5}_{n}6\underbrace{11\ldots1}_{n}, \quad n = 0, 1, 2, \ldots . \qquad (6)$$

Note that sequence (6) is not a part of OEIS® – the On-line Encyclopedia of Integer Sequences (www.oeis.org). This allows one to extend the list of the sequences of triangular numbers with amazing properties apparently not known before and pose the following problem

Prove that sequence (6) is the sequence of triangular numbers. Find the ranks of those numbers. Write down the closed and recursive formulas for sequence (6).

Posing such kinds of TITE problems enhances the existing mathematics curriculum. It gives strong justification to the use of technology in the classroom and, more generally, facilitates mathematics teaching and learning. Not less important is the fact that technology can serve as an agency for collateral learning in mathematical explorations.

Input:

Table$\left[k\,(10^n - 1)\left(\frac{1}{162}\,(k\,(10^n - 1) + 9)\right), \{k, 10\}, \{n, 10\}\right]$

Result:

{{1, 66, 6216, 617 716, 61 732 716, 6 172 882 716, 617 284 382 716,
61 728 399 382 716, 6 172 839 549 382 716, 617 283 951 049 382 716},
{3, 253, 24 753, 2 469 753, 246 919 753, 24 691 419 753, 2 469 136 419 753,
246 913 586 419 753, 24 691 358 086 419 753, 2 469 135 803 086 419 753},
{6, 561, 55 611, 5 556 111, 555 561 111, 55 555 611 111, 5 555 556 111 111,
555 555 561 111 111, 55 555 555 611 111 111, 5 555 555 556 111 111 111},
{10, 990, 98 790, 9 876 790, 987 656 790, 98 765 456 790, 9 876 543 456 790,
987 654 323 456 790, 98 765 432 123 456 790, 9 876 543 210 123 456 790},
{15, 1540, 154 290, 15 431 790, 1 543 206 790, 154 320 956 790,
15 432 098 456 790, 1 543 209 873 456 790, 154 320 987 623 456 790,
15 432 098 765 123 456 790}, {21, 2211, 222 111, 22 221 111,
2 222 211 111, 222 222 111 111, 22 222 221 111 111, 2 222 222 211 111 111,
222 222 222 111 111 111, 22 222 222 221 111 111 111},
{28, 3003, 302 253, 30 244 753, 3 024 669 753, 302 468 919 753,
30 246 911 419 753, 3 024 691 336 419 753, 302 469 135 586 419 753,
30 246 913 578 086 419 753}, {36, 3916, 394 716, 39 502 716,
3 950 582 716, 395 061 382 716, 39 506 169 382 716, 3 950 617 249 382 716,
395 061 728 049 382 716, 39 506 172 836 049 382 716},
{45, 4950, 499 500, 49 995 000, 4 999 950 000, 499 999 500 000,
49 999 995 000 000, 4 999 999 950 000 000, 499 999 999 500 000 000,
49 999 999 995 000 000 000}, {55, 6105, 616 605, 61 721 605,
6 172 771 605, 617 283 271 605, 61 728 388 271 605, 6 172 839 438 271 605,
617 283 949 938 271 605, 61 728 395 054 938 271 605}}

Figure 3. Generating a family of sequences of triangular numbers.

7. Conceptualizing the development of sequence (2)

As triangular numbers have the form $n(n + 1)/2$, one can check to see that the general term of sequence (2), $5\underbrace{00\ldots0}_{k}5\underbrace{00\ldots0}_{k}$, where $k = 0, 1, 2, \ldots$, can be presented as half the product of two consecutive integers. We have

$$5\underbrace{00\ldots0}_{k}5\underbrace{00\ldots0}_{k} = 5\cdot 1\underbrace{00\ldots0}_{k}1\underbrace{00\ldots0}_{k} = 5\cdot(10^{2k+1}+10^{k})$$
$$= 5\cdot 10^{k}(10^{k+1}+1) = \frac{1}{2}\cdot 10^{k+1}(10^{k+1}+1)\,.$$

As 10^{k+1} and $10^{k+1}+1$ are consecutive integers, the k-th term of sequence (2) is the triangular number of rank 10^{k+1}, $k = 0, 1, 2, \ldots$ and the terms x_k of sequence (2) can be generated through the formula

$$x_k = \frac{1}{2} \cdot 10^{k+1}(10^{k+1}+1), k = 0, 1, 2, \ldots . \tag{7}$$

Formula (7) implies that $x_0 = 55$ and it enables an equivalent representation of sequence (2) through the following recursive relation

$$x_{k+1} = x_k + 4950 \cdot 10^{2k} + 45 \cdot 10^k,\ \ x_0 = 55,\ \ k = 0, 1, 2, \ldots . \tag{8}$$

The transition from closed formula (7) to recursive formula (8) can be confirmed through the use of *Maple* as shown in Figure 4.

$P := k \to 0.5 \cdot 10^{k+1}(10^{k+1}+1)$

$$k \to 0.5\ 10^{k+1}\left(10^{k+1}+1\right)$$

$recur := P(k+1) - P(k)$

$$0.5\ 10^{k+2}\left(10^{k+2}+1\right) - 0.5\ 10^{k+1}\left(10^{k+1}+1\right)$$

$simplify(\%)$

$$4950.0\ 100^k + 45.0\ 10^k$$

Figure 4. *Maple*-based transition from (7) to (8).

Furthermore, using a spreadsheet (or any other computational tool) one can check to see that

$$\sqrt{1+8\times 55} = 21, \sqrt{1+8\times 5050} = 201, \sqrt{1+8\times 500500} = 2001,$$

that is, the square test holds true for the first three terms of (2), and then conjecture that

$$\sqrt{1 + 8 \cdot 5\underbrace{00\ldots0}_{k}5\underbrace{00\ldots0}_{k}} = 2\underbrace{00\ldots0}_{k}1 \quad . \tag{9}$$

In order to prove relation (9), note that $2\underbrace{00\ldots0}_{k}1 = 2 \cdot 10^{k+1} + 1$. Furthermore,

$$1 + 8 \cdot 5\underbrace{00\ldots0}_{k}5\underbrace{00\ldots0}_{k} = 1 + 8 \cdot (5 \cdot 10^{2k+1} + 5 \cdot 10^k) = 1 + 8 \cdot 5 \cdot 10^k(10^{k+1}+1)$$
$$= 1 + 4 \cdot 10^{k+1} \cdot (10^{k+1}+1).$$

Therefore, one has to prove the relation

$$1 + 4 \cdot 10^{k+1} \cdot (10^{k+1}+1) = (2 \cdot 10^{k+1}+1)^2 . \tag{10}$$

A simple transformation of both sides of relation (10) yields an obvious identity

$$1 + 4 \cdot 10^{2k+2} + 4 \cdot 10^{k+1} = 4 \cdot 10^{2k+2} + 4 \cdot 10^{k+1} + 1$$

which completes proof of relation (9).

It follows from (5) that

$$\frac{-1 + \sqrt{1 + 8 \cdot 5\underbrace{00\ldots0}_{k}5\underbrace{00\ldots0}_{k}}}{2} = \frac{-1 + 2\underbrace{00\ldots0}_{k}1}{2} = 10^{k+1}$$

indicating, once again, that all the terms of sequence (2) are triangular numbers the ranks of which are powers of ten; therefore, formula (7) can be used to generate the terms of sequence (2). The next step towards a TE component of exploring sequence (2) is to search it in the OEIS®. By entering a few first terms of the sequence into the search box of the OEIS® one can find out that a recursive formula for sequence (2) has the form

$$x_{n+1} = 110x_n - 1000x_{n-1}, x_0 = 1, x_1 = 55\,. \tag{11}$$

One can use *Maple* (Figure 5) to verify that sequence (11) is indeed a recursive formulation of sequence (2) which closed form was found to satisfy formula (7).

$P1 := n \rightarrow 0.5 \cdot 10^{n+1}\left(10^{n+1}+1\right)$

$$n \rightarrow 0.5\, 10^{n+1}\left(10^{n+1}+1\right)$$

$recur := P1(n) - 110 \cdot P1(n-1) + 1000 \cdot P1(n-2);$

$$0.5\, 10^{n+1}\left(10^{n+1}+1\right) - 55.0\, 10^{n}\left(10^{n}+1\right) + 500.0\, 10^{n-1}\left(10^{n-1}+1\right)$$

$simplify(recur)$

$$0.$$

Figure 5. Using *Maple* in demonstrating the equivalence of (7) and (11).

Input:

$\text{Table}\left[\frac{1}{2}(m \times 10^k)(m \times 10^k + 1), \{m, 9\}, \{k, 10\}\right]$

Result:

```
{{55, 5050, 500 500, 50 005 000, 5 000 050 000, 500 000 500 000, 50 000 005 000 000,
   5 000 000 050 000 000, 500 000 000 500 000 000, 50 000 000 005 000 000 000},
  {210, 20 100, 2 001 000, 200 010 000, 20 000 100 000,
   2 000 001 000 000, 200 000 010 000 000, 20 000 000 100 000 000,
   2 000 000 001 000 000 000, 200 000 000 010 000 000 000},
  {465, 45 150, 4 501 500, 450 015 000, 45 000 150 000,
   4 500 001 500 000, 450 000 015 000 000, 45 000 000 150 000 000,
   4 500 000 001 500 000 000, 450 000 000 015 000 000 000},
  {820, 80 200, 8 002 000, 800 020 000, 80 000 200 000,
   8 000 002 000 000, 800 000 020 000 000, 80 000 000 200 000 000,
   8 000 000 002 000 000 000, 800 000 000 020 000 000 000},
  {1275, 125 250, 12 502 500, 1 250 025 000, 125 000 250 000,
   12 500 002 500 000, 1 250 000 025 000 000, 125 000 000 250 000 000,
   12 500 000 002 500 000 000, 1 250 000 000 025 000 000 000},
  {1830, 180 300, 18 003 000, 1 800 030 000, 180 000 300 000,
   18 000 003 000 000, 1 800 000 030 000 000, 180 000 000 300 000 000,
   18 000 000 003 000 000 000, 1 800 000 000 030 000 000 000},
  {2485, 245 350, 24 503 500, 2 450 035 000, 245 000 350 000,
   24 500 003 500 000, 2 450 000 035 000 000, 245 000 000 350 000 000,
   24 500 000 003 500 000 000, 2 450 000 000 035 000 000 000},
  {3240, 320 400, 32 004 000, 3 200 040 000, 320 000 400 000,
   32 000 004 000 000, 3 200 000 040 000 000, 320 000 000 400 000 000,
   32 000 000 004 000 000 000, 3 200 000 000 040 000 000 000},
  {4095, 405 450, 40 504 500, 4 050 045 000, 405 000 450 000,
   40 500 004 500 000, 4 050 000 045 000 000, 405 000 000 450 000 000,
   40 500 000 004 500 000 000, 4 050 000 000 045 000 000 000}}
```

Figure 6. Using *Wolfram Alpha* in generating sequence (12).

Likewise, triangular numbers the ranks of which are multiples of the powers of ten can be generated. To this end, one can generalize formula (7) to the form

$$x_{k,m} = \frac{1}{2} \cdot m \cdot 10^{k+1}(m \cdot 10^{k+1} + 1), k = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots \quad (12)$$

Note that TE component of a problem may include the use of different software tools. For example, a spreadsheet is capable of displaying correctly a number with at most 15 digits. Once again, a tool that can complement a spreadsheet is *Wolfram Alpha*. Figure 6 shows the use of the tool in generating sequence (12) for $m = 1, 2, \ldots, 9$.

8. From modeling sequence (12) to exploring the sum of digits concept

One can observe in the data shown in Figure 6 that whatever the value of k, the sums of digits of the terms of sequence (12) are the same for a given value of m. The last observation can be explained by using different means, e.g., base-ten blocks. Also, noting that the sum of digits, *SD*, of an m-multiple of ten is equal to the sum of digits of the number m, that is, $SD(m \cdot 10^k) = SD(m)$, and that $SD(5m^2 \cdot 10^{2k-1} + 5m \cdot 10^{k-1}) = SD(5m^2 \cdot 10^{2k-1}) + SD(5m \cdot 10^{k-1})$, assuming that the number of digits, *ND*, of $5m$ is at least not greater than $2k - 1$, one can write

$$SD[5m \cdot 10^{k-1}(m \cdot 10^k + 1)] =$$
$$SD(5m^2 \cdot 10^k \cdot 10^{k-1} + 5m \cdot 10^{k-1}) = SD(5m^2) + SD(5m).$$

For example, for $m = 3$ we have $ND(3) = 1$ and the inequality $1 \le 2k - 1$ implies $k \ge 1$. So, when $m = 3$ and $k = 1$ we have

$$SD(5m^2 \cdot 10^{2k-1} + 5m \cdot 10^{k-1}) = SD(450 + 15) = SD(450) + SD(15)$$
$$= 9 + 6 = 15;\ SD(450 + 15) = SD(465) = 15.$$

At the same time, for $m = 11$ we have $ND(11) = 2$ and the inequality $2 \le 2k - 1$ yields $k \ge 1.5$; that is, $k > 1$ in order for

$$SD(5m^2 \cdot 10^{2k-1} + 5m \cdot 10^{k-1}) = SD(5m^2 \cdot 10^{2k-1}) + SD(5m \cdot 10^{k-1}).$$

Indeed, when $m = 11$ and $k = 1$ we have

$$SD(5m^2 \cdot 10^{2k-1} + 5m \cdot 10^{k-1}) = SD(6050 + 55) = SD(6105) = 12,$$

yet $SD(6050) + SD(55) = 11 + 10 = 21$. On the other hand, for $m = 12$ and $k = 1$ we have $SD(7200 + 60) = SD(7260) = SD(7200) + SD(60) = 15$. The spreadsheet shown in Figure 7 is designed to test whether the sum of digits function when applied to a sum of two numbers yields the same result when it is applied to each of the addends separately. In this investigation, the $SD(N)$ is calculated through the formula

$$SD(N) = N - 9 \cdot \sum_{i=1}^{n} INT\left(\frac{N}{10^i}\right) \quad (13)$$

where *INT* is the greatest integer function and $n = ND(N) - 1$. In the spreadsheet of Figure 7 the triples of rows (6, 7, 8), (9, 10, 11), …, (18, 19, 20), corresponding to the values of $k = 1, 2, \ldots, 5$, include the values of $5m^2 \cdot 10^{2k-1}$, $5m \cdot 10^{k-1}$, and $5m^2 \cdot 10^{2k-1} + 5m \cdot 10^{k-1}$ with the corresponding values of m displayed in row 1. Then, by using formula (13), the equality $SD(5m^2 \cdot 10^k \cdot 10^{k-1} + 5m \cdot 10^{k-1}) = SD(5m^2) + SD(5m)$ is tested and when it does not hold true, the number 1 is displayed in row 22. The computations confirm that the last equality does not hold true only in the case $k = 1$ when $ND(5m) \le 2$.

	A	B	C	L	M	N	O	P	Q	R	S	T
1	m	1	2	11	12	13	14	15	16	17	18	19
2	5m	5	10	55	60	65	70	75	80	85	90	95
3	5m^2	5	20	605	720	845	980	1125	1280	1445	1620	1805
4												
5	k											
6	1	50	200	6050	7200	8450	9800	11250	12800	14450	16200	18050
7	1	5	10	55	60	65	70	75	80	85	90	95
8	1	55	210	6105	7260	8515	9870	11325	12880	14535	16290	18145
9	2	5000	20000	605000	720000	845000	980000	1125000	1280000	1445000	1620000	1805000
10	2	50	100	550	600	650	700	750	800	850	900	950
11	2	5050	20100	605550	720600	845650	980700	1125750	1280800	1445850	1620900	1805950
12	3	500000	2000000	60500000	72000000	84500000	98000000	112500000	128000000	144500000	162000000	180500000
13	3	500	1000	5500	6000	6500	7000	7500	8000	8500	9000	9500
14	3	500500	2001000	60505500	72006000	84506500	98007000	112507500	128008000	144508500	162009000	180509500
15	4	50000000	200000000	6050000000	7200000000	8450000000	9800000000	11250000000	12800000000	14450000000	16200000000	18050000000
16	4	5000	10000	55000	60000	65000	70000	75000	80000	85000	90000	95000
17	4	50005000	200010000	6050055000	7200060000	8450065000	9800070000	11250075000	12800080000	14450085000	16200090000	18050095000
18	5	5000000000	20000000000	6.05E+11	7.2E+11	8.45E+11	9.8E+11	1.125E+12	1.28E+12	1.445E+12	1.62E+12	1.805E+12
19	5	50000	100000	550000	600000	650000	700000	750000	800000	850000	900000	950000
20	5	5000050000	20000100000	6.05001E+11	7.20001E+11	8.45001E+11	9.80001E+11	1.125E+12	1.28E+12	1.445E+12	1.62E+12	1.805E+12
21												
22	1			1		1		1		1		1
23	1											
24	1											
25	2											
26	2											
27	2											
28	3											
29	3											

Figure 7. Using a spreadsheet in exploring the sum of digits function.

More specifically, one can see that what was described for m = 11 continues for a few other odd values of m. Likewise, what was observed for m = 12 continues for other even values of m. This shows the complexity of the sum of digits concept as one attempts to generalize and it may serve as a source of further investigations into the ideas described in this section. That is, a TE part of a problem can then be followed by its TI part, which may be rather challenging. One can see how the exploration of sequence (2), or any other computer-based mathematical exploration can be organized as a continuous juxtaposition of argument and computation.

9. A direct approach to exploring sequence (1)

Consider now sequence (1). Computing

$$\sqrt{1+8\cdot 21}=13, \sqrt{1+8\cdot 2211}=133, \sqrt{1+8\cdot 222111}=1333,$$

may prompt one to conjecture that in general

$$\sqrt{1+8\cdot \underbrace{22\ldots 2}_{k}\,\underbrace{11\ldots 1}_{k}}=1\,\underbrace{33\ldots 3}_{k} \tag{14}$$

The proof of formula (14) can be construed as a TITE problem. It is more complicated in comparison with the proof of (4) and it can be carried out with the help of technology. That is, at that point one makes a decision about the need to computationally support a pure mathematical argument – a TI component of the problem. To this end, the numbers involved in (14) have to be represented in terms of the powers of ten. Indeed,

$$1 + 8 \cdot \underbrace{22 \ldots 2}_{k}\,\underbrace{11 \ldots 1}_{k} =$$

$$1 + 8 \cdot [2 \cdot (10^{2k-1} + 10^{2k-2} + \cdots + 10^{k}) + (10^{k-1} + 10^{k-2} + \cdots + 1)]$$

$$= 1 + 8 \cdot [2 \cdot 10^{k} \cdot (10^{k-1} + 10^{k-2} + \cdots + 1) + (10^{k-1} + 10^{k-2} + \cdots + 1)] =$$

$$1 + 8 \cdot (10^{k-1} + 10^{k-2} + \cdots + 1)(2 \cdot 10^{k} + 1) =$$

$$1 + 8 \cdot (2 \cdot 10^{k} + 1) \cdot \sum_{i=0}^{k-1} 10^{i}\ .$$

Also,

$$1\,\underbrace{33 \ldots 3}_{k} = 10^{k} + 3 \cdot (10^{k-1} + 10^{k-2} + \cdots + 1) = 10^{k} + 3 \cdot \sum_{i=0}^{k-1} 10^{i}.$$

Therefore, one has to prove that

$$1 + 8 \cdot (2 \cdot 10^{k} + 1) = (10^{k} - 1) \sum_{i=0}^{k-1} 10^{i} = (10^{k} + 3 \cdot \sum_{i=0}^{k-1} 10^{i})^{2}. \qquad (15)$$

Noting that $\sum_{i=0}^{k-1} 10^{i} = (10^{k} - 1)/9$, the last relation can be re-written as

$$1 + (8/9) \cdot (2 \cdot 10^{k} + 1)(10^{k} - 1) - (1/9) \cdot (4 \cdot 10^{k} - 1)^{2} = 0 \qquad (16)$$

and its symbolic proof can be outsourced to *Wolfram Alpha* or *Maple*. Figure 8 shows the use of *Wolfram Alpha* in proving identity (16).

Alternatively, proof of relation (15) can be carried out by *Maple* without replacing sums with closed representations. Figure 9 (a TE component of the problem) shows the proof of (15). To this end, one can define two functions: $P1(k)$ – the left-hand side of (15) and $P2(k)$ – the right-hand side of (15), and then simplify the difference $P1(k) - P2(k)$ by using symbolic computational capability of *Maple* to show that it is equal to zero.

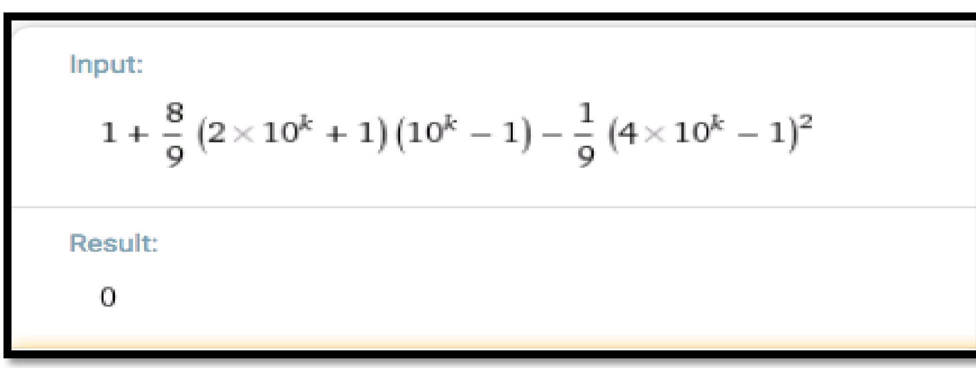

Figure 8. *Wolfram Alph*a calculates the left-hand side of (16).

$P1 := n \rightarrow 1 + 8 \cdot \left(2 \cdot 10^{n} + 1\right) \cdot sum\left(10^{i}, i = 0..n - 1\right)$

$$n \rightarrow 1 + \left(16\, 10^{n} + 8\right) \left(\sum_{i=0}^{n-1} 10^{i} \right)$$

$P2 := n \rightarrow 3 \cdot\ sum\left(10^{i}, i = 0..n - 1\right) + 10^{n}$

$$n \rightarrow 3 \left(\sum_{i=0}^{n-1} 10^{i} \right) + 10^{n}$$

$differ := P1(n) - (P2(n))^2$

$$1 + \left(16\, 10^{n} + 8\right) \left(\frac{1}{9}\, 10^{n} - \frac{1}{9} \right) - \left(\frac{4}{3}\, 10^{n} - \frac{1}{3} \right)^2$$

$simplify(\%)$

$$0$$

Figure 9. *Maple*-based proof of relation (15).

Finally, due to the relations

$$\frac{-1+13}{2} = 6, \frac{-1+133}{2} = 66, \frac{-1+1333}{2} = 666, \ldots, \frac{-1+1\underbrace{33\ldots3}_{k}}{2} = \underbrace{66\ldots6}_{k}$$

one can conclude that the numbers 21, 2211, 222111, 22221111, ... $\underbrace{22\ldots2}_{k}\underbrace{11\ldots1}_{k}$ are triangular numbers the ranks of which are, respectively, 6, 66, 666, ..., $\underbrace{66\ldots6}_{k}$. The following TITE problem can be formulated: *How can one generate computationally triangular numbers the ranks of which consist of the digit six only?*

To begin with the TI component of the formulated problem, note that $\underbrace{66\ldots6}_{k} = 6 \cdot \underbrace{11\ldots1}_{k}$. Therefore, 11 = 10 + 1, 111 = 100 + 11, 1111 = 1000 + 111 and so on. In general, $\underbrace{11\ldots1}_{k} = 10^{k-1} + \underbrace{11\ldots1}_{k-1}$. In other words, the recursive formula $x_k = x_{k-1} + 10^{k-1}, x_1 = 1$ defines the terms of the sequence $x_k = \underbrace{11\ldots1}_{k}$.

Alternatively, we have $11 = 99 \div 9 = (10^2 - 1) \div 9; 111 = 999 \div 9 = (10^3 - 1) \div 9$ and, in general, $x_k = \underbrace{11\ldots1}_{k} = (10^k - 1) \div 9$. That is,

$$y_k = \frac{6x_k(6x_k+1)}{2} = 3 \cdot \frac{10^k - 1}{9}\left(6 \cdot \frac{10^k-1}{9} + 1\right) = \frac{10^k-1}{3}\left(\frac{2(10^k-1)}{3} + 1\right) = \frac{(10^k-1)(2 \cdot 10^k + 1)}{9},$$

or

$$y_k = \frac{(10^k-1)(2 \cdot 10^k + 1)}{9} \qquad (17)$$

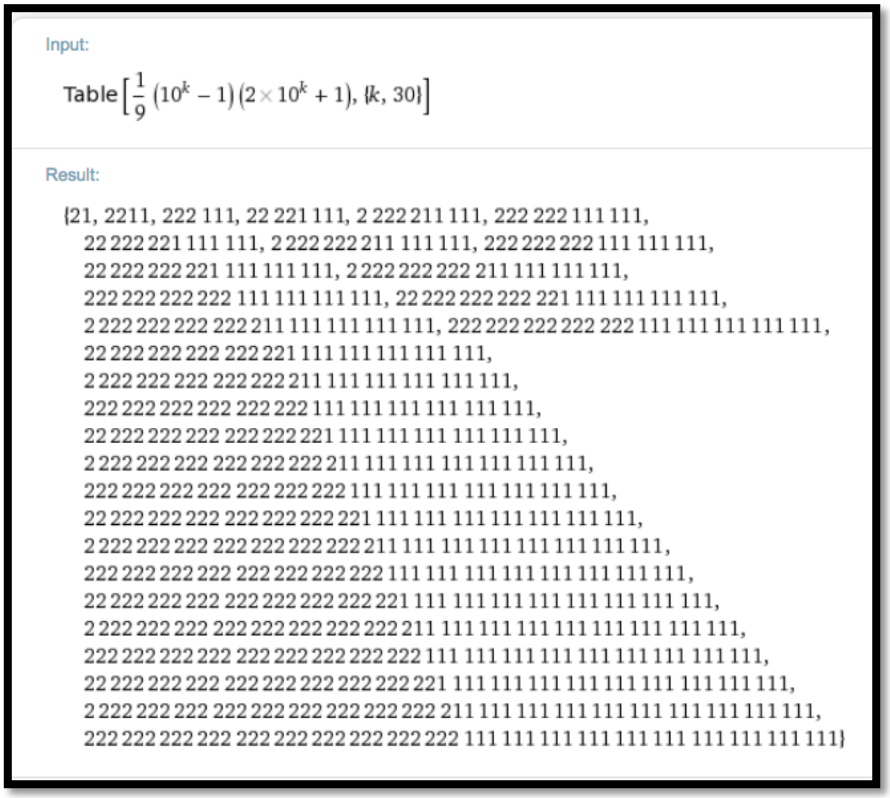

Figure 10. Using *Wolfram Alpha* in generating sequence (17) up to a 60-digit term.

Formula (17) is a closed formula for sequence (1). Figure 10 shows the use of *Wolfram Alpha* in generating sequence (1) through formula (17). Note that *Wolfram Alpha* has clear advantage over a spreadsheet as the former tool is capable of generating integers with a large number of digits.

10. Posing new TITE problems

Consider the sequence

$$45, 2415, 224115, 22241115, 2222411115, \dots . \qquad (18)$$

The common term of (18) has the form $N_n = \underbrace{22 \ldots 2}_{n-1}4\underbrace{11 \ldots 1}_{n-1}5$ – a $2n$-digit number, where n = 1, 2, 3, …., so that when n = 1 we have $N_1 = 45$.

A TI part of exploring sequence (18) is to develop its closed formula. To this end, one can write

$$N_n = \underbrace{22 \ldots 2}_{n-1} 4 \underbrace{11 \ldots 1}_{n-1} 5 = 2 \cdot \underbrace{11 \ldots 1}_{n-1} \cdot 10^{n+1} + 4 \cdot 10^n + \underbrace{11 \ldots 1}_{n-1} \cdot 10 + 5$$

$$= \underbrace{11 \ldots 1}_{n-1} \cdot 10(2 \cdot 10^n + 1) + 4 \cdot 10^n + 5$$

$$= 10(10^{n-2} + 10^{n-3} + \cdots + 1)(2 \cdot 10^n + 1) + 4 \cdot 10^n + 5 =$$

$$= (10^{n-1} + 10^{n-2} + \cdots + 10)(2 \cdot 10^n + 1) + 4 \cdot 10^n + 5$$

$$= \frac{10^n - 10}{9} (2 \cdot 10^n + 1) + 4 \cdot 10^n + 5.$$

That is, each term of sequence (18) can be written in the form

$$\frac{10^n - 10}{9} (2 \cdot 10^n + 1) + 4 \cdot 10^n + 5.$$

A TE part of exploring sequence (18) is to use *Wolfram Alpha* in solving the equation

$$\frac{x^2 + x}{2} = \frac{10^n - 10}{9} (2 \cdot 10^n + 1) + 4 \cdot 10^n + 5.$$

The program yields the positive root of this equation, $x = \frac{2 \cdot 10^n + 7}{3}$. This indicates that all the terms of sequence (17) are triangular numbers of rank $\frac{2 \cdot 10^n + 7}{3}$. From here, a number of TITE problems can be formulated.

1. Prove the identity

$$\frac{2 \cdot 10^n + 7}{6} \cdot \left(\frac{2 \cdot 10^n + 7}{3} + 1\right) = \frac{10^n - 10}{9} (2 \cdot 10^n + 1) + 4 \cdot 10^n + 5.$$

2. Prove that the sum $2 \cdot 10^n + 7$ is a multiple of 9 for any natural number *n*. Use different methods of proof: a) test of divisibility by 9; b) method of mathematical induction; c) technology-based proof.

3. Show that the sequence $x_n = \frac{2 \cdot 10^n + 7}{3}$ generates the numbers

$$9, 69, 669, 6669, \ldots, \underbrace{666 \ldots 6}_{n-1}9, \ldots \qquad (19)$$

where *n* = 1, 2, 3, … . In other words, prove the identity $\underbrace{666 \ldots 6}_{n-1}9 = \frac{2 \cdot 10^n + 7}{3}$.

Note that whereas sequences (18) and (19) do not belong to OEIS®, the sum of digits of the terms of sequences (18) and (19) are, respectively, 9, 12, 15, 18, … and 9, 15, 21, 27, … have an interesting relationship: the latter consists of every second term of the former. The same relationship can be observed for the terms of sequence (1) and their ranks. Indeed, the sum of digits of sequence (1) are consecutive multiples of three and their ranks are even multiples of three. This property, while can be easily explained as the sum of digits of the terms of sequences (18) and (19) increases by three and six, respectively, it is not a trivial one: already the sums of digits of the terms of sequence (2) are all equal to ten, but the sums of digits of their ranks are all equal to one.

11. Conclusion

This paper introduced teaching ideas born recently in connection with the need to develop a coherent problem-solving mathematics curriculum. The coherence of the curriculum requires from mathematics educators an ability to balance positive and negative affordances of technology in the lieu of modern technological advances capable of reducing problem solving to students' pushing right buttons on a keyboard without conceptual understanding of mathematics involved. Towards this end, the concept of a TITE problem was discussed. The discussion stemmed from revisiting earlier uses of technology in exploring sequences of numbers that represent subsequences of triangular numbers the digits of which follow interesting patterns. In a more general context, the concept of a TITE problem-solving curriculum extends the notion of Type II application of technology [23] to address the issue of a negative affordance of computers when a mathematical problem may be educationally appropriate, yet could be solved through a pure computer routine. The didactic idea behind addressing this issue is to use such an easy approach to mathematical problem solving as a motivation for developing new, more demanding tasks which require both a non-artificial mathematical thinking and computational support. Such uses of technology in mathematical problem solving can be referred as its Type II applications of the second order.

At the beginning of the 21st century, reflecting on the use of computers in mathematics education, Langtangen and Tveito [21] argued, "Much of the current focus on algebraically challenging, lengthy, error-prone paper and pencil work can be significantly reduced. In fact, we seriously doubt that there will be space for this type of activity at all in a few decades, at least not in the mainstream education. The reason for such an evolution is that the computer is simply much better than humans on any theoretically phrased well-defined repetitive operation" (pp. 811-812). Technology was also suggested as an assistant in doing proof by professional mathematicians who would be using technology, as a means "to put the correctness of their proofs beyond reasonable doubt" [18, p. 1405]. This perspective on the use of technology is not only applicable to students majoring in mathematics. The general mathematics education system should take these particular recommendations into account and make appropriate changes in the curriculum in order to provide all learners of mathematics with a new type of problems, technology immune/technology enabled (TITE) ones, and to suggest pedagogically-sound ways of outsourcing symbolic computations to software.

The idea of using TITE problems in mathematics teacher education can be seen as an interaction between their TI and TE components in much the same way as the concept of TPCK (technological pedagogical content knowledge) provides teacher candidates with "understanding of the interactions of the knowledge of technology and the knowledge of their subject area" [28, p. 520]. With this in mind, the importance of conceptual knowledge of mathematical structures was emphasized in this paper through the use of numeric sequences the development of which is visually obvious but structurally challenging. In order to move from visual recognition to conceptual understanding, one has to learn how to look at a problem through the lenses of the TITE concept. Such learning is not an easy task for it requires multiple cognitive skills to come into play. Those skills include knowing when technology is an appropriate scaffold in the process of entertaining mathematical reasoning and what type of technology has to be used in support of which kind of mathematical thinking. In presenting these teaching ideas, several learning frameworks have been mentioned and brought together to form a unifying environment for the learning of mathematics with computers by secondary teacher candidates. The authors believe that mathematics educators working for an EPP (education preparation provider) unit can develop many extensions of the activities presented in the paper to be used by the candidates in posing and solving new TITE problems.

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References

[1] Abramovich, S. (2014). *Computational Experiment Approach to Advanced Secondary Mathematics Curriculum*. Dordrecht, The Netherlands: Springer.

[2] Abramovich, S. (2015). Educating teachers to pose mathematical problems in the digital age: toward alternative ways of curriculum design. *Open Mathematical Education Notes*, **5**(2): 115-136.

[3] Abramovich, S. (2015). Mathematical problem posing as a link between algorithmic thinking and conceptual knowledge. *The Teaching of Mathematics*, **18**(2): 45-60.

[4] Abramovich, S., and Connell, M. L. (2015). Digital fabrication and hidden inequalities: connecting procedural, factual, and conceptual knowledge. *International Journal of Technology for Teaching and Learning*, **11**(2): 76-89.

[5] Abramovich, S., and Connell, M. L. (2017). TITE problem solving: integrating computing and proving in secondary mathematics teacher education. In *Proceedings of Society for Information Technology & Teacher Education International Conference 2017* (pp. 1321-1327). Chesapeake, VA: Association for the Advancement of Computing in Education.

[6] Abramovich, S., Fujii, T., and Wilson, J. (1995). Multiple-Application Medium for the Study of Polygonal Numbers. *Journal of Computers in Mathematics and Science Teaching*, **14**(4): 521-557.

[7] Angeli, C., and Valanides, N. (2009). Epistemological and methodological issues for the conceptualization, development and assessment of ICT-TPCK: Advances in technological pedagogical content knowledge (TPCK). *Computers & Education*, **52**(1): 154-168.

[8] Australian Association of Mathematics Teachers. (2006). *Standards for Excellence in Teaching Mathematics in Australian Schools* [On-line materials]. Available at:www.aamt. edu.au.

[9] Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M. and Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress, *American Educational Research Journal*, **47**(1): 133–180.

[10] Beiler, A. H. (1966). *Recreations in the Theory of Numbers*. New York: Dover.

[11] Common Core State Standards. (2010). *Common Core Standards Initiative: Preparing America's Students for College and Career* [On-line materials]. Available at: http://www.corestandards.org.

[12] Conference Board of the Mathematical Sciences. (2012). *The Mathematical Education of Teachers II*. Washington, DC: The American Mathematical Society.

[13] Connell, M. L., and Abramovich, S. (2017). Stamping Functions: A Technology Immune Technology Enabled Problem within an Action on Objects Framework. In *Proceedings of Society for Information Technology & Teacher Education International Conference 2017* (pp. 1328-1334). Chesapeake, VA: Association for the Advancement of Computing in Education.

[14] Council for the Accreditation of Educator Preparation. (2013). *CAEP Accreditation Standards and Evidence: Aspirations for Educator Preparation*. Recommendations from the CAEP Commission on Standards and Performance Reporting to the CAEP Board of Directors. Washington, DC: Author.

[15] Dahlstrom, E., and Bichsel, J. (2014). *ECAR Study of Undergraduate Students and Information Technology, 2014. Research report*. Louisville, CO: ECAR, October 2014. Available at http://www.educause.edu/ecar.

[16] Department for Education (2013). *National Curriculum in England: Mathematics Programmes of Study*. Crown copyright. Available at: https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study.

[17] Dewey, J. (1938). *Experience and Education*. New York: MacMillan.

[18] Harrison, J. (2008). Formal proof — theory and practice. *Notices of the American Mathematical Society*, **55**(11): 1395-1406.

[19] International Society for Technology in Education. (2008). *National Educational Technology Standards and Performance Indicators for Teachers*. Eugene, OR: Author.

[20] Kennedy, G.E., Judd, T.S., Churchward, A., Gray, K., and Krause, K.L. (2008). First year students' experiences with technology: Are they really digital natives? *Australasian Journal of Educational Technology*, **24**(1): 108-122.

[21] Langtangen H. P., and Tveito, A. (2001). How should we prepare the students of science and technology for a life in the computer age? In B. Engquist and W. Schmid (Eds.), *Mathematics Unlimited—2001 and Beyond* (pp. 809-825). New York: Springer.

[22] Mishra, P., and Koehler, M. J. (2006). Technological pedagogical content knowledge. *The Teachers College Record*, **108**(6): 1017-1054.

[23] Maddux, C. D. (1984). Educational microcomputing: The need for research. *Computers in the Schools*, **1**(1): 35-41.

[24] Ministry of Education, Singapore. (2006) *Secondary Mathematics Syllabuses,* Curriculum Planning and Development Division: Author. Available at: http://www.moe.edu.sg/education/syllabuses/sciences/files/maths-secondary.pdf.

[25] Ministry of Education, Singapore. (2012) *N(T)-level Mathematics Teaching and Learning Syllabus*, Curriculum Planning and Development Division: Author. Available at: http://www.moe.gov.sg/education/syllabuses/sciences/files/normal-technical-level-maths-2013.pdf.

[26] National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.

[27] National Curriculum Board. (2008). *National Mathematics Curriculum: Framing Paper*. Australia: Author. Available at http://www.ncb.org.au/verve/_resources/ National_Mathematics_Curriculum_-_Framing_Paper.pdf.

[28] Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, **21**(5): 509-523.

[29] Pólya, G. (1954). *Induction and Analogy in Mathematics*. Princeton, NJ: Princeton University Press.

[30] Prensky, M. (2001). Digital natives, digital immigrants. *On the Horizon*, **9**(5): 1–6.

[31] President's Council of Advisors on Science and Technology. (2010). *Prepare and Inspire: K-12 Education in Science, Technology, Engineering and Math (STEM) for America's Future*. Washington, DC. Author. Available at: https://www.whitehouse.gov/sites/default/files/microsites/ostp/pcast-stemed-report.pdf.

[32] Takahashi, A., Watanabe, T., Yoshida, M., and McDougal, T. (2006). *Lower Secondary School Teaching Guide for the Japanese Course of Study: Mathematics (Grade 7-9). Madison, NJ: Global Education Resources.*

[33] Western and Northern Canadian Protocol. (2008). *The Common Curriculum Framework for Grades 10-12 Mathematics*. Available at http://www.bced.gov.bc.ca/irp/pdfs/mathematics/WNCPmath1012/2008math1012wncp_ccf.pdf.

[34] Youngman, C. E. (1905). Problem 15648. *Mathematical Questions and Solutions from Educational Time*s. New Series, vol. **8**: 33-34.