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Pre-Service Mathematics Teacher Identity Issues with Respect to Learning and Teaching High School Geometry

Jacqueline Sack
University of Houston–Downtown, United States
sackj@uhd.edu

Judith Quander
University of Houston–Downtown, United States
quanderr@uhd.edu

Laura Mitchell
University of Houston–Downtown, United States
mitchelll@uhd.edu

Abstract. In order to bring attention to the incoherence of traditional pre-high-school geometry curricula, the first two authors have developed a course to provide secondary pre-service mathematics teachers a robust understanding of how geometry could be learned. Using the Van Hiele levels as a guiding framework, the course uses isometric transformations as tools to generate and then define geometric concepts of the important triangles and quadrilaterals. Descriptions of key activities from the course are provided. Initially, their second cohort of pre-service teachers did not grasp the purpose of these approaches, fully expecting to be doing the same kind of geometry they had experienced in high school. When expected to use this detailed knowledge to create their own deductive proofs the point was made. The authors use an identity framework to analyze their students' evolving identities, that changed from compliance for the purpose of passing to total immersion in the entire course.

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1. Introduction

Texas is one of the many states in the United States that struggles to fill secondary mathematics teaching positions each year and schools often fill these positions with teachers who are certified to teach in other areas [1] or not certified at all. These shortages are most pronounced in high-needs schools, such as those in urban areas like Houston. These schools have difficulty finding and then retaining high-quality teachers [4] especially in fields like mathematics. The University of Houston–Downtown (UHD) is an urban institution in the middle of downtown Houston and makes a priority to prepare students to address the needs of the greater Houston community. Preparing teachers who will succeed in urban schools is part of that priority. UHD is a commuter, urban institution with an enrollment of over 14,000 [10] and is a federally recognized Hispanic and minority serving institution with a large number of African-American students. In addition, UHD has a large non-traditional student population.

The UHD mathematics department, supports an average total of 82 total majors per year. They are generally strong as indicated by a 93% course completion rate and that 52% have at least a 3.0 (B) grade point average [9] UHD has the potential to make a contribution to the number of well-qualified high school mathematics teachers in the region. In the fall of 2011, we established the University of Houston-Downtown Noyce Mathematics Teacher Scholarship Program to recruit and financially support strong undergraduate mathematics majors who were seeking secondary mathematics teacher certification. The program was funded by the National Science Foundation Noyce Teacher Scholarship grant [6]. Able to provide substantial scholarships to these students, hereafter called *Noyce scholars*, four faculty members (two are authors of this paper), three from the mathematics department and one from the education department, have worked closely to support student success as undergraduates and later, as beginning high school teachers. Through this grant, the education department co-principal investigator, Sack, has used her course release funds to co-teach an upper-level semester-long course, Geometry for High School Teachers, with PI, Quander, attended by all Noyce scholars as well as other non-Noyce undergraduate students majoring or minoring in mathematics. This article describes the rationale for the course, including its theoretical underpinnings; the instructors' expectations and perspectives on student interactions during class meetings; and, one activity that appeared to change the dynamics of a recent course section during one particular class meeting in interesting ways. The authors used an identity focus to frame their perspectives on student identity with respect to learning geometry.

2. Identity framework

We frame our identity construct about *normative* and *personal* identity described by Cobb, Gresalfi, & Hodge [2]. They denote *normative identity* as that established in the classroom by the instructor and as enacted by the learners in order to be viewed by the instructor as an effective doer of the mathematics in that classroom. Learners would have to identify with these expectations. Normative identity is a communal rather than individualistic construct. In a learner-centered classroom, the instructor provides learners many opportunities to authorize decisions about the interpretation of tasks sometimes by carefully guiding an exploration and asking guiding questions to enable learners to develop conceptual meanings and understandings. In such classrooms, learners exercise "*conceptual agency*, which involves choosing methods and developing meanings and relations between concepts and principles" [2, p. 45]. In contrast, in classrooms in which learners use only prescribed, established solution methods, they exercise *disciplinary agency*. In such classrooms, learning authority rests almost entirely with the teacher.

There is a continuum between conceptual versus disciplinary agency in that teachers may provide learners some opportunity to exercise conceptual agency, but also expect their mathematical productions to follow traditionally acceptable formats. The authors, as instructors, lean toward learner-centered instruction to enhance learners' conceptual agency with respect to developing geometric sense, but also expect them to exercise procedural agency in producing mathematically acceptable written arguments, especially in writing geometric proofs.

Personal identities, on the other hand, concern the extent to which individual students "identify with their classroom obligations, merely cooperate with the teacher, or resist in engaging in classroom activities" [2, p. 47].

In the next section, the theoretical backdrop that was used to develop the course trajectory, *The Van Hiele Model of Geometric Thought* [11] is described with its connections to beginning course activities.

3. Theoretical backdrop for the course

The first author has developed a teaching-learning trajectory for high school geometry instructors that has evolved over 14 years, during the 6 years that she served as an instructional coach for high school teachers in a large urban school district, then as a mathematics methods instructor and researcher for the past 8 years. Since a large number of high school students lack the pre-requisite geometric knowledge to be successful in a traditional deductive geometry high school course, the trajectory builds this knowledge through carefully developed activities based on the Van Hiele Levels [11], prior to deductive proof activities. The author uses the names of the levels as presented in Van Hiele's primary source [11] rather than synonymous names from secondary sources. These are briefly described as follows:

Visual Level. Learners use informal language to describe a geometric figure by its appearance. This is appropriate for primary grades learners.

Descriptive Level. Learners define figures using formal language and geometric notation. Properties are determined by measurement or overlaying (to compare for congruence, for example) resulting in an exhaustive list of properties to define a given figure. It is appropriate to begin this type of work beginning in intermediate grades (approximately age 9).

Relational Level. Learners use logical language (not symbolic logic) and may be able to follow a simple proof but are unable to develop their own proofs. The relational level may be developed at the same time as the descriptive level, by comparing two related but different figures' properties.

Deductive Level. Learners can follow and create formal geometric proofs as expected in high school.

Rigor Level. This is typically post-high school deductive work, e.g. symbolic logic, truth tables.

Van Hiele [11] also proposed a sequence of phases to be followed in any given lesson. These align with inquiry-based, learner-centered instruction, beginning with introductory information that sets the stage for an investigation; followed by learner-centered exploration, which may be guided by the teacher to enhance learners' conceptual agency with respect to the topic at hand; then, a whole-class discussion about learners' findings; then, extended investigation or problem solving related to the initial task; and, finally, integration of knowledge generated through the entire lesson cycle. This may become the introductory information for the next lesson cycle.

In the United States, there is a strong emphasis on numerical and algebraic reasoning from primary grades through high school. A good example of widely adopted mathematics curriculum objectives is the Common Core Mathematics curriculum (http://www.corestandards.org/Math/). The geometry strand, however, is poorly developed with respect to the Van Hiele [11] levels across the elementary and middle grades, with a capstone single year deductive geometry course offered in high school. For example, in Kindergarten and 1st Grade, learners correctly name shapes, use informal language to describe numbers of vertices or sides (Visual and beginning Descriptive levels). In 2nd Grade, they recognize and draw shapes having specified attributes, such as number of angles in 2D figures, and to identify triangles, quadrilaterals, pentagons, hexagons and cubes (beginning Descriptive level). In 3rd Grade, they recognize that shapes may share attributes (e.g., rhombuses, rectangles, squares all have 4 sides), which is a Relational level task with very limited development of Descriptive level understanding of quadrilaterals to support this objective. In 4th Grade, learners draw and identify points, lines, rays, line segments, angles (acute, right, obtuse), and perpendicular and parallel lines in 2D figures. In 4th Grade, protractor measurement is introduced and mastered. These tasks are more closely aligned to developing Descriptive to Relational level understanding of 2D figures that should take place before the hierarchical categorization set for 3rd Grade. In 5th Grade, 2D figures are classified in a hierarchy based on properties. This could easily be done within the 4th Grade work if substantial Descriptive level work is included in 3rd and 4th Grades. In 6th Grade, there are no objectives relating geometric shapes. Only in 7th Grade learners draw, construct (using given angle or side measurements) and describe geometric figures and the relationships among them along with facts about supplementary, complementary, vertical and adjacent angles and algebraic opportunities to solve simple equations about angles in figures. In 8th Grade, by experimentation, they verify properties of isometric transformations, presumably in the coordinate plane, and also prove and apply the Pythagorean Theorem, which is a Deductive level concept that should be developed in the high school Geometry course. Unfortunately, most often, this theorem is introduced as an abstract formula that applies to right triangles with given measurements, totally devoid of conceptual development.

4. The Geometry for High School Teachers Course

The authors believe that teachers will use instructional resources appropriately only if they have experienced them as learners. Therefore, their learners are introduced to appropriate pre-high-school activities at the onset of this course, to be adequately prepared to address the conceptual gaps described above. It begins with a series of activities to develop intuitive Visual to Relational Levels understanding of isometric transformations, which are then used as tools to exhaustively develop the properties of parallel lines, triangles and quadrilaterals that should have been carefully developed in pre-high school mathematics classes.

4.1 ISOMETRIC TRANSFORMATIONS

Many of the activities utilize patty paper, which is inexpensive wax-free, transparent parchment paper procured through catering outlets. If unable to obtain the patty paper in 6-inch (approximately 15 cm) squares, then larger sheets can be cut to this size. It is not necessary that these be perfect squares. Under the instructor’s direction, learners use the patty paper to construct all of the figures themselves using the properties they develop about isometric transformations. In the first activity, they fold the paper in half, draw an asymmetric figure on one side of the fold (the pre-image), and trace the figure on the other side (the image), forming a reflection, or mirror image. This is a Visual Level construction. Each learner uses his or her own figure to make generalizations that follow. Next, each learner draws the line segments connecting three or four corresponding points from the pre-image to the image across the fold, which is the line of reflection. Now, they describe the reflection in terms of these line segments in relation to the line of reflection, which brings their understanding to the Descriptive Level with respect to the concept of reflection (see Fig. 1). Even though learners started with very different self-constructed figures, the Descriptive Level structure and properties are the same for all.

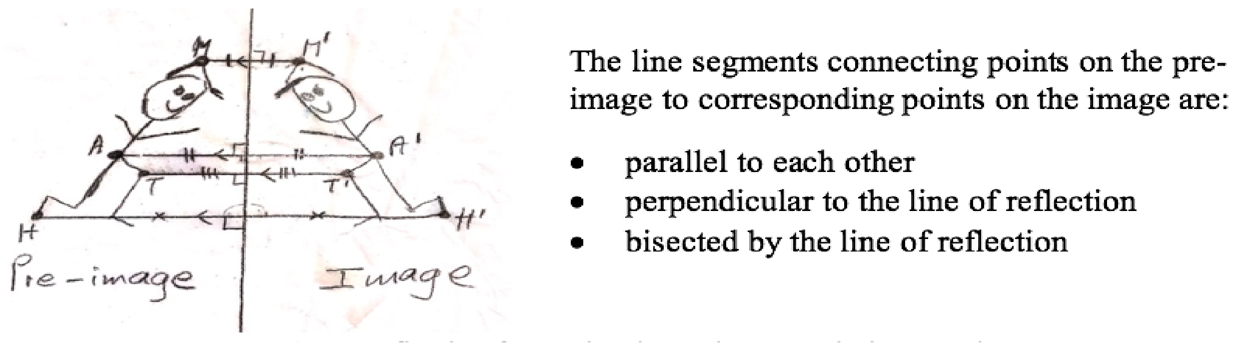


Figure 1. Reflection from Visual Level to Descriptive Level

For the next activity (see Fig. 2), learners create two parallel fold lines, about 2 inches (5 cm) apart, draw an asymmetric figure (pre-image) on one side of the parallel fold lines, reflect it by folding and tracing into the middle section (image-1), and finally reflecting image-1 across the second fold by tracing into the third section (image-2). As before, they draw line segments connecting at least three points on the pre-image through their corresponding points on image-1 to those on image-2. Using the knowledge of properties of reflection derived from the first activity, the properties of translation are developed. Thus, Descriptive Level information about translations emerge from Relational Level thinking using their prior knowledge of reflection.

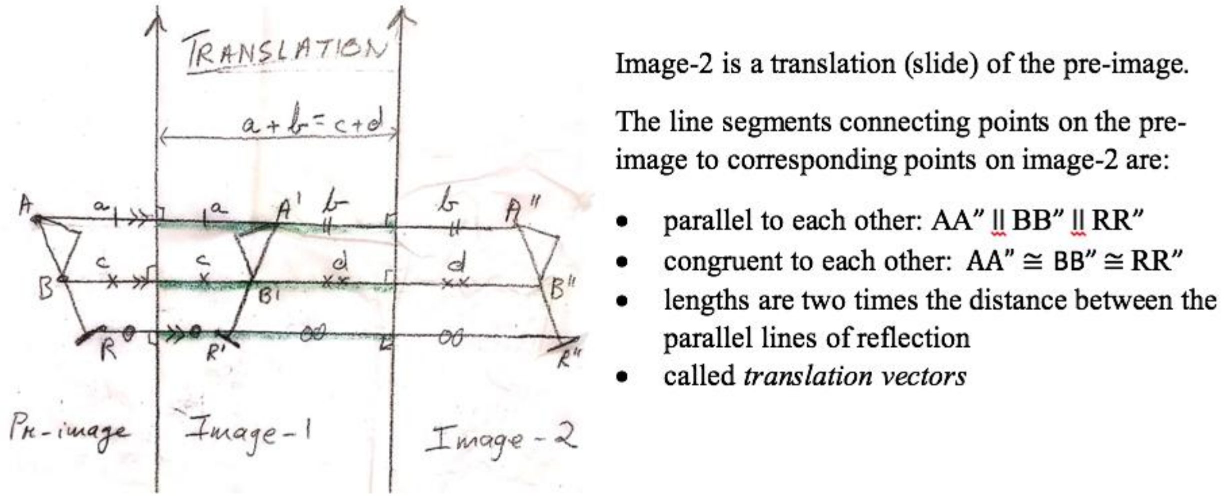


Figure 2. Translation from Descriptive to Relational Level

A third activity involves reflection across two intersecting lines of reflection, using perpendicular folds (to generate a half-turn, 180-degree rotation or point-reflection); and, also across two fold lines that intersect in any random angle to generate a rotation of the pre-image through two times the angle formed by the folds about the intersection point generated by the two fold lines (see Fig. 3).

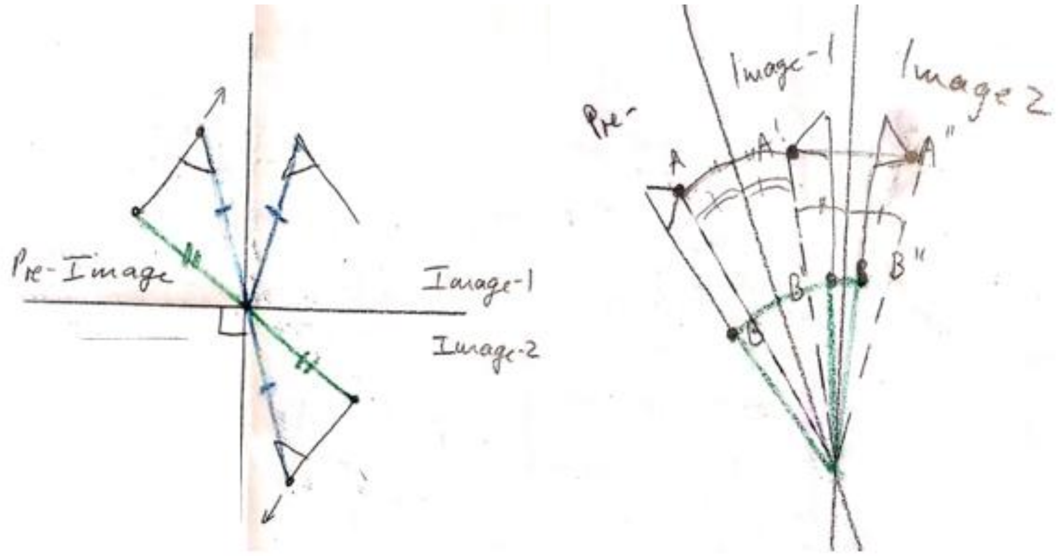


Figure 3. Rotation integrating Descriptive and Relational Level

In this course, Relational Level understanding of isomeric transformations underscores the development of Descriptive Level and Relational Level understanding of triangles, parallel line properties and the special quadrilaterals. Of note: The Relational Level with respect to the isometric transformations, as experienced above, is often introduced only within the last quarter of traditional high school text books, but can easily be done with children in 4th Grade concurrently with angle measurement.

4.2 WHAT MAKES A TRIANGLE?

Many learners who are still at the Visual Level across many geometric concepts believe that any set of three side measurements can create a triangle. For the first triangle activity, each learner draws three randomly selected cards from a packet with cards numbered from 3 to 30 and then uses only a ruler to construct a triangle using a self-selected unit of measurement for all three sides. Note that compass constructions will be introduced later in the course when a Relational Level understanding of rhombus and kite properties are in place. The instructor writes the headings $a \leq b \leq c$ and *Triangle? Yes/No* on the board. Learners list their three numbers in increasing order underneath *a*, *b*, and *c*, and Y if they were able to construct a triangle; N if they were unable to construct a triangle. This activity leads to the *triangle inequality* postulates: *The sum of the lengths of any two sides of a triangle is always greater than the length of the third side*, and, *The smallest angle is opposite the shortest side; the largest angle is opposite the longest side*. Again, through the process of constructing these figures based on the three selected random numbers, all learners wrestled with the task and were all able to make sense of which sets could form a triangle.

Learners now draw small acute or obtuse scalene triangle on an index card. Using three different colored pencils the sides and angles are marked to clearly distinguish them. They cut out the triangle, and then tessellate it on a sheet of plain paper by tracing the triangle and keeping the colored markings visible at all times. The only rule is to completely match congruent sides of the traced triangles; no gaps or overlaps allowed. This process, using rotation produces all of the angle properties related to intersecting lines, parallel lines and transversals, and that the sum of the three angles in a triangle is always 180^{o}.

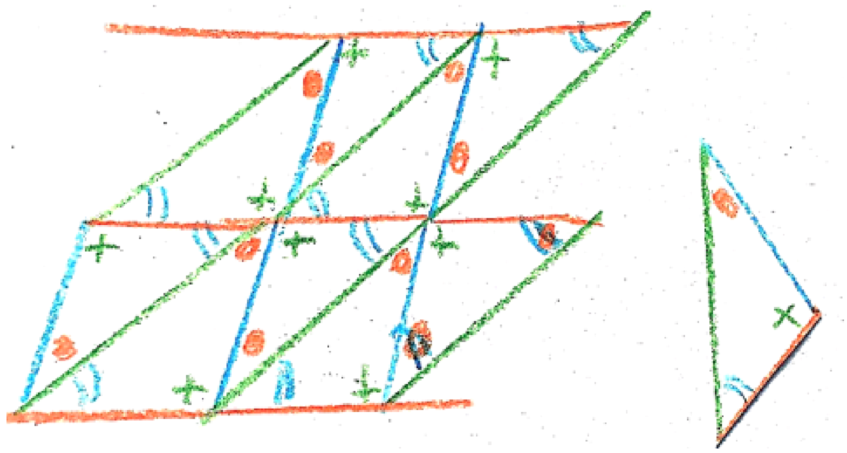
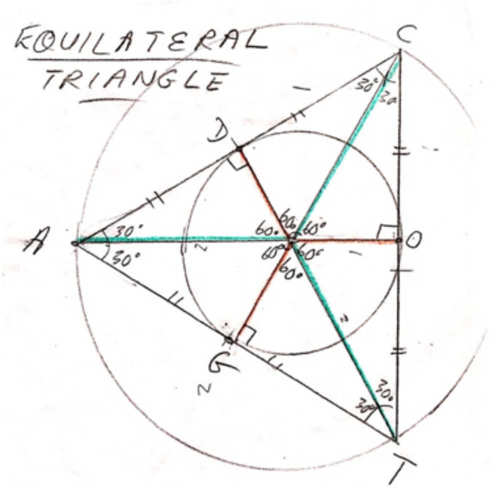


Figure 4. Triangle tessellation with associated angle properties

In the next activity, each learner constructs an equilateral triangle on patty paper, having drawn one side, approximately 4 inches (10 cm) in length, near one of the edges of the paper. The instructor encourages learners to try various ways to construct the figure, so that when an efficient method is presented it will be understood. Inevitably, some of the learners will notice that the third vertex point must lie immediately above the midpoint of the drawn line (i.e., on its perpendicular bisector. This is constructed by folding the patty paper so that the two endpoints of the drawn line coincide. Now the drawn line can be folded through one of its endpoints so that the other endpoint coincides with the perpendicular bisector fold line. All the properties of an equilateral triangle are now visible and an exhaustive list of properties is listed (see Fig. 5).



- three congruent 60° angles
- three congruent sides
- three lines of symmetry that create
  - 6 large 30-60-90 right triangles
  - 6 small 30-60-90 right triangles
  - 3 30-120-30 isosceles obtuse triangles
  - 3 kites
- the common point of intersection is the incenter and the circumcenter of the triangle; and divides each symmetry segment in a 2:1 ratio
- 3-fold or 1/3-turn symmetry

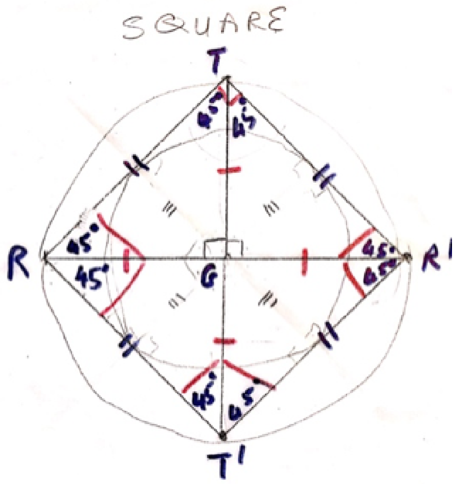
Figure 5. Equilateral triangle and properties

Additional triangle construction activities ensue. These include development of properties of acute, right and obtuse isosceles triangles, and, the four points of concurrency, namely the circumcenter, incenter, centroid and orthocenter; and, the Euler Line, which passes through the circumcenter, centroid and orthocenter. These activities are developed using patty paper and properties of reflection. For the activity to construct isosceles acute, obtuse and right triangles first draw three congruent circles on three sheets of patty paper. If compasses are not handy or learners struggle to use compasses, any circular object with a 2-3 inch diameter can be traced. The centers of these circles can be found by folding the circle into two semi-circles and pinching the approximate center of the diameter – and then repeated along a different fold so that the pinched folds intersect in the center of the circle. The isosceles triangle properties can be constructed using two radii on any size circle to form their legs. For our activity, the three different isosceles triangles are constructed on congruent circles, and then properties relating to central angles and chord lengths can also be determined. Details are available on the course website [8].

Van Hiele [11] had emphasized first examining figures with the most symmetries in a given set. For example, the equilateral triangle has 3 lines of symmetry (see Figure 5) and will therefore display more properties (Descriptive Level) than the non-equilateral isosceles triangles, which have only 1 line of symmetry. Learners can develop properties for any other triangle by considering which properties of the equilateral triangle apply to these triangles (Relational Level).

4.3 QUADRILATERALS

When moving to quadrilaterals, the course also begins with the most symmetrical, i.e., the square. Each learner constructs a square as follows. First, fold a sheet of (square-shaped) patty paper into two sections, then fold a perpendicular fold that will divide the paper into four quadrants. In one quadrant, an isosceles right triangle is drawn, using the perpendicular fold lines to place the right angle. The first leg, along one of the folds can be matched by folding to accurately place the endpoint of the congruent second leg. The properties of this triangle are marked clearly, using conventional symbols such as angle measures and matching tick marks for the congruent segments. This triangle is then reflected across one of its legs; and then the entire figure is reflected across the line containing the other leg (see Fig. 6). The figure is seen to be a square, since it has 4 right angles and 4 congruent sides. Now all of the properties of a square can be determined in terms of its sides, vertex angles, diagonals and symmetry. Some of these are listed in Figure 5, and it is possible to expand the list to over 30 properties.



Sides
4 congruent sides
adjacent sides perpendicular
opposite sides parallel

Diagonals
congruent to each other
perpendicular to each other
bisect each other
bisect the vertex angles
lie on 2 lines of symmetry

Vertex Angles
4 congruent right angles
opposite angles congruent
sum of angles = 360°

Symmetry
4 lines of symmetry:
2 contain the diagonals
2 lie on the perpendicular bisectors of the side
2-fold and 4-fold rotational symmetry

Figure 6. Properties of a square

Again, even though this activity is teacher-directed, each learner has constructed his or her own square on patty paper using properties of reflection. These properties then allow learners to determine the properties of the figure exhaustively (Descriptive Level). In following constructions, using reflection or rotation, each figure's properties are determined by comparison with those of the square.

Figure 7 shows the patty-paper constructions of a rhombus (by reflecting a scalene right triangle across the lines containing its legs, like the square's construction); a rectangle (by rotating a right scalene triangle about its hypotenuse (half-turn or 180^{O} rotation); a parallelogram (by rotating a scalene acute or obtuse triangle about one of its sides); and a kite (by reflecting a scalene triangle across one of its sides – but, in the case of a right triangle, reflecting across its hypotenuse). All of the above activities were designed to explicate the properties (through Descriptive and Relational Level) of the important 2-D figures that high school learners use in Deductive Level proof activities. Unfortunately, the pre-high school geometry curriculum does not move coherently to build robust Relational Level knowledge of these figures, that is the cornerstone for success at the Deductive Level. Some additional activities engage learners in deeper Relational Level understanding among the different triangles and quadrilaterals that they now understand in exhaustive detail.

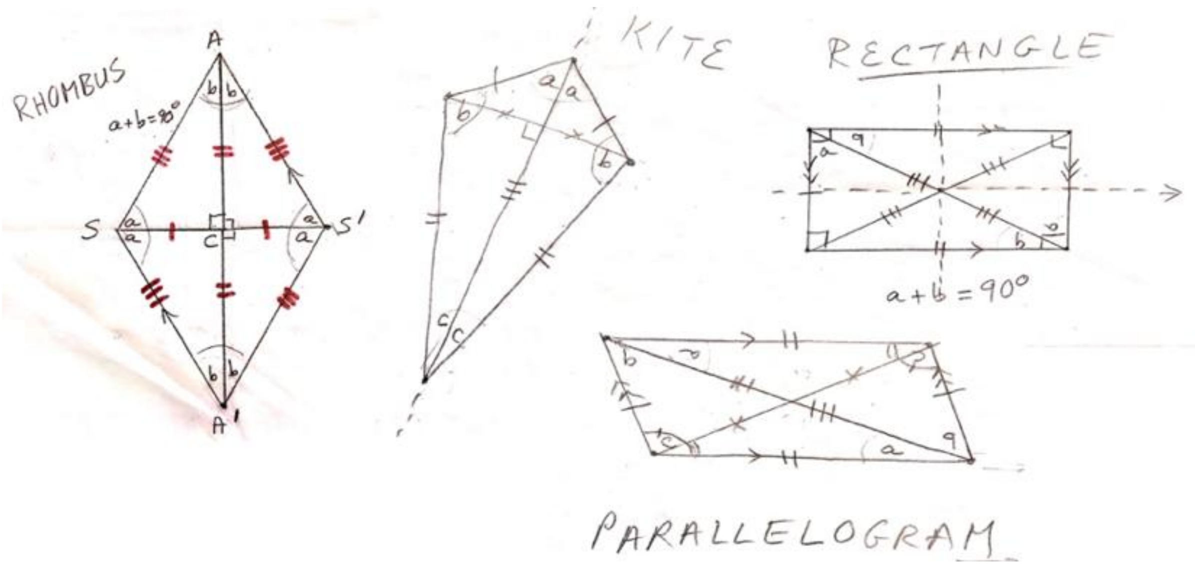


Figure 7. Rhombus, Rectangle, Parallelogram and Kite

4.4 DEDUCTIVE PROOFS

The next section of work moves learners into Deductive Level proof activities. It begins with an understanding of the Euclidean postulates for triangle congruence, namely, which three corresponding congruent parts of two triangles make the triangles congruent? Consider all combinations: side – side – side; side – included angle – side; angle – included side – angle; and, angle – angle – non-included side. The quadrilaterals are then defined using conventional conditional statements, such as, *If a quadrilateral is a rhombus then it has four congruent sides*. Using these definitions, prove that the properties previously developed in the Descriptive/Relational Level activities, are true using triangle congruence. These proofs are typical of those found in traditional high school texts.

5. Classroom dynamics

Students who major or minor in mathematics select this Geometry for High School Teachers course because they have a genuine interest in becoming secondary mathematics teachers or because they need an upper-level mathematics elective course for their particular degree plans. Therefore, none of them would resist engaging in classroom activities since their grades depended on their successful classroom participation and the quality of their application of this knowledge in course exams. From a personal identity perspective, they would at least cooperate and complete required assignments even if they did not engage with them meaningfully. The instructors had co-taught this course once before, three semesters prior to the current experience. Their first class of 20-25 students eagerly participated in all activities from the onset, displaying strong personal identity relations with the way their geometric understanding evolved. As a result, several who had not committed to becoming teachers at the beginning of that semester enrolled in the university's secondary education program, increasing the number of Noyce scholars from approximately 2 per semester at the same stage of their program to 12, in that particular semester. A very strong majority of students in that particular cohort had engaged in all course activities very enthusiastically and never missed a class. As a result, the few who were merely complying with course expectations did not voice or demonstrate their perspectives vocally but did have more erratic attendance patterns.

We expected this second cohort of 20-25 students to be much like the first. However, most of them began to show the same erratic attendance as the small minority had in the first course section. They still complied and completed course assignments, being careful to send these to us if they knew they would be absent. Furthermore, there was a distinct lack of enthusiasm for responding to our expectation that they would share their thinking about the geometric concepts emerging from their personal construction experiences. Only 2 or 3 students consistently engaged in vocalizing their ideas. We frequently asked them to work in pairs to develop ideas about properties of figures they had individually constructed before whole-class discussion took place. Each pair then shared one, or occasionally, two ideas in order to comply with our normed expectations of participation in concept development, but without the energy and excitement we had enjoyed with the first group. When we recorded their ideas on the board, we also noted that many students in this class would move to the front to take pictures with their smart phones. Therefore, they clearly demonstrated their intent to cooperate and obtain the information they would need to pass the class, but they did not identify strongly with the learner-centered conceptual agency the instructors had hoped for.

Then, a remarkable change in classroom engagement and focus occurred as we moved to using the triangle congruence properties to perform Deductive Level proofs. Having a list of the properties that traditionally define each quadrilateral (e.g., If a quadrilateral has four right angles then it is a rectangle), in pairs, they were to select no more than two properties (not the traditional defining properties) from a given quadrilateral and try to prove that the defining properties were true. It was at this particular juncture in the course that both instructors noticed a radical change in learner engagement, which led them to look at their own and their learners' identities with respect to learning and teaching high school geometry.

As the class began to work on this activity, the two instructors moved about the room, quietly working with each pair of students on their selected problem. They noticed a dramatic change in the whole class' level of participation. It had changed from low energy compliance to an enthusiastic buzz. For example, one particular pair of students decided to focus on rectangles. They wanted to begin with congruent diagonals and believed that this would be sufficient to justify the figure to be a rectangle. The instructor working with them suggested they try to draw counter-examples, thinking that if they were not able to do so and this property was convincingly sufficient, then the next step would be to set up the steps to the formal proof. They were very uneasy with this request, and so the instructor sketched a figure with congruent diagonals that intersected at non-equivalent points along each diagonal, and deliberately not perpendicular to each other. They immediately realized they would need another property that would force the figure to be a rectangle, and eagerly referenced the exhaustive, Descriptive Level list of properties the class had generated for this figure. They selected, congruent diagonals that bisect each other, and worked out the proof. They were so enthusiastic about their success that, after having their work checked, they immediately set about selecting a different pair of properties to sufficiently define a rectangle.

Cobb, Gresalfi, and Hodge (2009) describe the process of personal identification to include turning from "obligations-to-others" into "obligations-to-oneself" (p. 47). We realized that this activity had set up conditions for this to occur. The following week, we asked our students to anonymously respond to an activity that would help them and ourselves make sense of their identity awareness with respect to learning and teaching geometry. We provided a list of items along with the option of responding on a *Cluster Write* (Rico 2000) to guide and elaborate on their thinking (see Table 1). They were provided the list of items but did not necessarily respond to these in any order or place on the cluster-write sheet. Students wrote comments on a page with "Geometry for High School Teachers" circled in the middle of the page. See Fig. 8 for an example cluster write from this class.

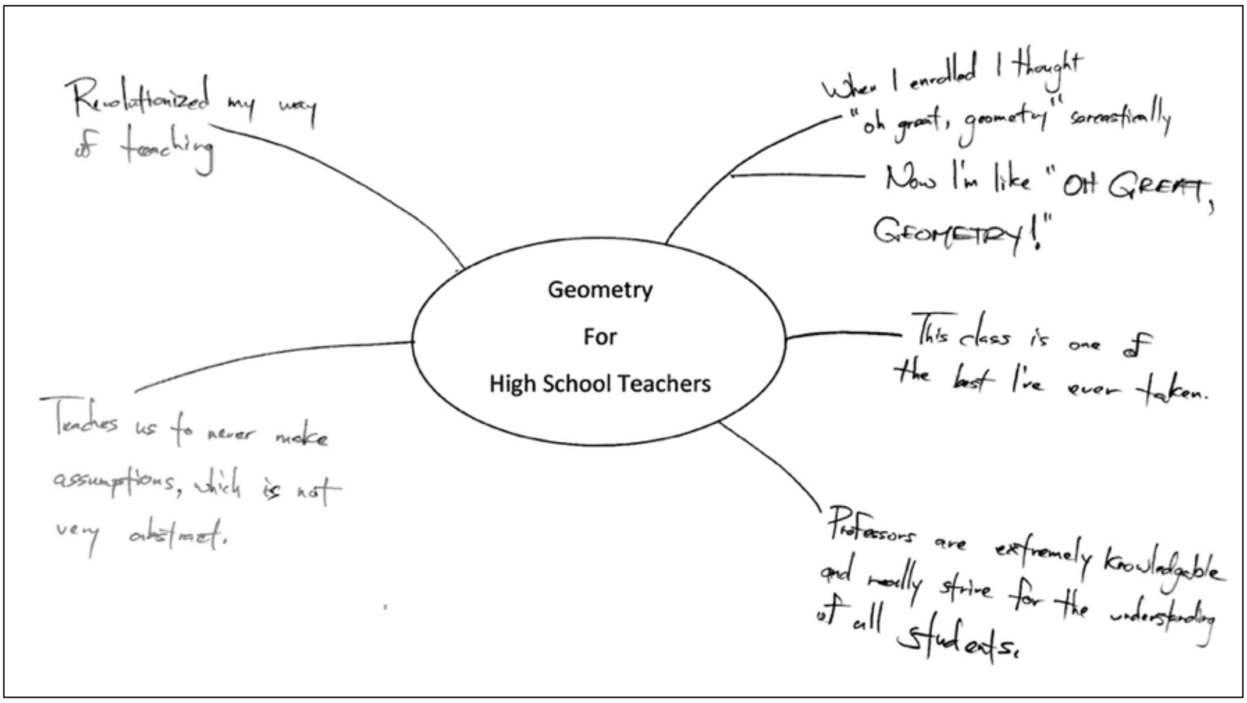


Figure 8. Example cluster write

6. Results

The three authors, including a third researcher who was uninvolved in teaching the course but who has a strong background in identity research, independently identified themes emerging from the cluster writes during the analysis of the data. The set of items and selected responses are listed in Table 1.

1. **Why did you choose to take this class?**

   [Most responded that it was on their degree plans.]

2. **What were your expectations for this class when you first started?**
   a. To love geometry because I honestly hate this subject.
   b. Expected it to be rigorous and boring.
   c. I thought it would be easier – algebraic geometry.
   d. I hoped to pass this course.
   e. I expected to review high school material.
   f. When enrolled, I thought, “oh great, geometry” – sarcastically
   g. Excited about taking it (yep, still am). Material would just be review: Review, yes, but it had more added.
   h. My expectations were to simply return all the stuff they taught me in high school geometry.

3. **How did the class line up with your expectations in the first few weeks?**
   a. I got a chance to have my mind refilled with this knowledge and I still learned new things about old topics.
   b. (from “expected it to be rigorous and boring”): Nothing at all what I expected.
   c. So far so good, I knew it would be different and it has been.
   d. Pretty right on. My only concerns have been: will what we are learning here line up with what my future school is doing. How they are teaching things?
   e. I was confused but now everything is coming together.
   f. We learned a different and better style than what I was taught in jr. high.
   g. First couple of weeks – very confused.
   h. I expected to review high school material – first few weeks OK, learned basics, but then a lot more was added– so it kinda threw me off b/c a different approach was taken.

4. **What has or has not worked well for you in this class?**
   a. My grade – somehow I feel I understand the material but my grade says otherwise. Knowledge-wise I’m happy.
   b. Love the patty paper. Hate proof.
   c. Performing exercises with patty paper helps my understanding but as soon as we went to proofs it was a shockwave of confusion.
   d. Has given me different approaches to teach the subject areas.
   e. Talking about what I do or do not understand.
   f. It takes me out of my comfort zone.
   g. Where are the grids? (possible reference to “algebraic geometry.”)
   h. Patty paper exercises – The best for this course! – definitely recommend.
   i. Now, I’m like “Oh great, GEOMETRY!”
   j. Definitely happy taking this course
   k. I’ve really enjoyed this class. The way the information flows and builds on itself was very helpful in understanding the concepts.
   l. It was not as boring as I thought it would be. I learned more than just geometry, but also how to teach it in a way that is more sensible and fun.

5. **Share your thoughts about last week’s proof experiences (selecting two properties to sufficiently define the quadrilateral through deductive proof).**
   a. It was fun and great to see how different aspects can work together and prove the activities. It also helped me connect the dots with what we have been working with.
   b. I still struggle with proofs but the more exposure the more I understanding. I enjoyed and was surprised on how many ways it could be proven.
   c. It is fun because it helps us think where things (formulas / definitions) come from. What we

thought we knew finally makes sense.

d. That was very good and helped a lot.
e. The last class intrigued me because it made us think outside of the box that what we were used to. It was another way to prove things we already knew using a less direct path. It put things in a different perspective.
f. Proofs helped put the properties together.
g. Happy! Wonderful! Finally I have learned some rules about geometry.
h. I learned to look beyond the box.
i. The proofs scared me – I had no sense of how to start the proof. Later, I understood more than before but I don't know if it's too late.
j. The proof topic has been one of the only enjoyable classes.
k. Last week's proof experience was incredible insightful. I felt that I learned more about doing proofs than I have in all my math classes doing proofs. Knowing already about the quadrilaterals really helps to do the proofs.

Table 1. Cluster Write Items and Typical Responses

Note: Even though 18 students participated and all 18 sets of responses were analyzed, the authors culled responses down to those that directly responded to each given prompt. They noted a number of responses to be irrelevant to given prompts.

7. Discussion

Item 2 intended to bring forth some sense of students' incoming identities with respect to learning and teaching geometry. Responses a, b, c, e, g, h indicate that students expected the course to be a repeat of high school geometry, but five of the six displayed a negative perspective. Only responses g and possibly h reflect positive identity with respect to the subject. One student expected the course to be "algebraic geometry," a term used to describe those high school geometry courses that reinforce algebraic skills throughout, for example, representing the figures on the *xy*-coordinate plane; focusing on slopes and segment lengths using the distance formula to elucidate properties of figures; performing transformations across the axes or about the origin; and, finding the values of angles in which algebraic expressions are used with respect to the angle properties in various figures.

Item 3 indicates that that initial expectations were for the course to be a repeat of high school geometry, correlating with a personal identity of cooperating with instructors to make an easy passing grade on material that was familiar. Six commented that they were learning new things, that two found to be confusing in the beginning but then began to make sense. Generally, they liked the different conceptually-based instructional approaches indicating a shift toward an identity that aligned more closely with the instructors' expectations. Response d reflects a strong alignment with the instructors' conceptual agency but concern about how they will be expected to teach in their future classrooms, likely to have a strong disciplinary agency as dictated by the curriculum and their future school district expectations.

Item 4 asks for specific positive and negative aspects of the class. Only two comments (f, g) indicate personal identities relating strongly to doing what is needed to pass the class. The other ten comments were all very positive and align with strong conceptual agency identity. These ten responses all show personal preference for the style of teaching and content that made the course relevant and exciting. Only one, f, indicated a desire to rehash the "algebraic geometry" so prevalent in high school geometry courses in the United States. Responses c, d, and l indicate learners' perspectives on the different ways that geometry can be taught.
Item 5 asked students to focus on the particular activity that the instructors noted to have moved students' personal identities toward more conceptual, instructor-aligned perspectives. All eleven

comments were positive and indicated that their identities had now completely aligned with instructors' intentions. No negative comments about this item were noted. Only one, i, shared personal concern about constructing proofs. Another, k, stated that the prior experiences in developing detailed knowledge about the different figures helped to do the proofs.

8. Conclusions

The instructors' overarching goal was to provide future high school mathematics teachers a perspective on learning and teaching geometry that prepares learners for Deductive Level proof work rather than the watered-down "fill in the blank" pre-prepared proofs and coordinate grid, "algebraic geometry" work so prevalent in high school texts across the United States. Item 2, response c, and item 4, response g, referring to "algebraic geometry" shows the typical high school perspective of using geometric figures on the coordinate plane to prepare learners for Pre-Calculus and Calculus where transformation of functions on the coordinate plane is important.

Considering that these students are majoring or minoring in mathematics one can reasonably expect that they enjoy doing mathematics. That the majority of them indicated an initial dislike for geometry, or the sense that they would find the course easy – as a repeat of high school geometry – indicates that we need to take a careful look at the K-8 and high school geometry curriculum in the United States. All responses in item 5 showed a change in our students' perspectives about teaching and learning geometry, and about the importance of the activities that established detailed knowledge of the figures they would encounter in appropriate deductive work.

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