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FROM PASCAL'S TRIANGLE TO A MATHEMATICAL FRONTIER: UNCOVERING HIDDEN PATTERNS IN THE OSCILLATIONS OF GENERALIZED GOLDEN RATIOS

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Abstract. This note demonstrates how, using commonly available computer applications, allows for the discovery of patterns that structure oscillations of generalized golden ratios formed by the roots of the so-called Fibonacci-like polynomials. It is shown how the universality of oscillations associated with the smallest root, regardless of the length of the string of the ratios, can be established and formulated in combinatorial terms. In addition, the use of circular diagrams informed by computational experiments in the case of the strings of odd number lengths made it possible to recognize patterns associated with the largest root.

Key words: Pascal's triangle, generalized golden ratios, cycles, oscillations, Fibonacci-like polynomials, technology, circular diagrams, permutations.

ZDM Subject Classification: I70, N70, R20

1. Introduction

This note has five major goals – mathematical, educational, and epistemological –listed as an introduction to this note. The first goal is to show how proceeding just from the well-known Pascal's triangle, an open mathematical problem formulated in very simple terms and motivated by computations only can be presented in the context of mathematics education. The second goal is to suggest that the notion of experimental mathematics [6] can be extended from the domain of research to the domain of didactics. The third goal is to highlight one of the most profound notions of mathematics epistemology that allows for a number of fundamental concepts of mathematics to become connected in a problem-solving setting. The fourth goal is to show how modern and classic mathematical ideas can be brought to bear when computational experiment is utilized as a signature pedagogy of mathematics. The fifth goal is to share ideas about using commonly available technology tools in the context of a capstone secondary mathematics education course to demonstrate to prospective teachers "how questions arising in high school can lead to the frontiers of current research" [9, p. 64].

2. From Pascal's triangle to Fibonacci-like polynomials

Consider one of the most famous mathematical structures, Pascal's triangle (Figure 1). According to Kline [12], Pascal came across his triangle through recording sample spaces of experiments of tossing n coins for different values of n. Using the letters H (head) and T (tail) to denote two possible outcomes resulted from tossing a coin, the following special cases can be recorded. When $n=1$, the sample space

$\Omega_1 = \{H, T\}$ can be associated with the string $(1,1)$ meaning that the set Ω_1 comprises one outcome with one head and one outcome with no head. When $n = 2$, the sample space $\Omega_2 = \{HH, HT, TH, TT\}$ can be associated with the string $(1,2,1)$ meaning that the set Ω_2 comprises one outcome with two heads, two outcomes with one head, and one outcome with no heads. When $n = 3$, the sample space $\Omega_3 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ can be associated with the string $(1,3,3,1)$ meaning that the set Ω_3 comprises one outcome with three heads, three outcomes with two heads, three outcomes with one head, and one outcome with no heads. One can see that, indeed, the above three strings of numbers are, respectively, the second, third, and fourth rows of Pascal's triangle.

1	↙	↙							
1	1	↙	↙						
1	2	1	↙	↙					
1	3	3	1	↙	↙				
1	4	6	4	1	↙				
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1

Figure 1. Pascal's triangle.

The entries of Pascal's triangle can be rearranged to turn the so-called shallow diagonals (pointed at by the arrows in Figure 1) into the rows of a new triangular-like array of numbers shown in Figure 2. Alternatively, this array can be developed from the first column by computing partial sums of units (to get the sequence of natural numbers in the second column), then by computing partial sums of natural numbers (to get the sequence of triangular numbers in the third column), then by computing partial sums of triangular numbers (to get tetrahedral numbers in the fourth column), then by computing partial sums of tetrahedral numbers in the fifth column (to get pentatope numbers in the sixth column), and so on.

It is well known that the sums of numbers in each row of the triangular-like array of Figure 2 are consecutive Fibonacci numbers. What was not known until recently [1-3] is that if one uses these strings of numbers as coefficients of polynomials by the powers of *x* so that the far-right number in each row is the coefficient in x^0 followed by the coefficient in x^1 and so on, the sequence of polynomials called Fibonacci-like polynomials [3]

$$P_0(x) = 1,\ P_1(x) = x+1,\ P_2(x) = x+2,\ P_3(x) = x^2+3x+1,\ P_4(x) = x^2+4x+3,$$

$$P_5(x) = x^3+5x^2+6x+1,\ P_6(x) = x^3+6x^2+10x+4, \ldots$$

each of which does not have complex roots, whatever the power of a polynomial is, can be developed. Note that $P_2(x) = P_1(x) + P_0(x)$, $P_3(x) = xP_2(x) + P_1(x)$, $P_4(x) = P_3(x) + P_2(x)$, $P_5(x) = xP_4(x) + P_3(x)$, $P_6(x) = P_5(x) + P_4(x)$, and, in general,

$$P_n(x) = x^{\mathrm{mod}(n,2)} P_{n-1}(x) + P_{n-2}(x),\ P_0(x) = 1,\ P_1(x) = x+1 \qquad (1)$$

where $\mathrm{mod}(n,2)$ is the remainder of *n* divided by 2. The remarkable property of a Fibonacci-like polynomial $P_n(x)$ to have exactly *n* real roots was established by the authors computationally for $n \le 100$ and, as of the time of writing this note, it remains a technology-motivated conjecture.

1							
1	1						
1	2						
1	3	1					
1	4	3					
1	5	6	1				
1	6	10	4				
1	7	15	10	1			
1	8	21	20	5			
1	9	28	35	15	1		
1	10	36	56	35	6		
1	11	45	84	70	21	1	
1	12	55	120	126	56	7	
1	13	66	165	210	126	28	1
1	14	78	220	330	252	84	8

Figure 2. Pascal-like triangle as a generator of Fibonacci-like polynomials.

Consider now the recursion equation

$$g_{k+1} = a + \frac{b}{g_k}, \quad g_1 = 1 \tag{2}$$

where *a* and *b* are real parameters, $a^2 + 4b < 0$. The significance of equation (2) is in its connection to the famous difference equation associated with Fibonacci numbers and the Golden Ratio. Indeed, setting $g_k = \frac{f_{k+1}}{f_k}$, $f_1 = f_2 = 1$, it follows from (2) that $f_{k+2} = af_{k+1} + bf_k$ and when $a = b = 1$ the celebrated numbers 1, 1, 2, 3, 5, 8, ... and the Golden Ratio, $\lim\limits_{k\to\infty} \frac{f_{k+1}}{f_k} = \frac{1+\sqrt{5}}{2}$, result. When $a^2 + 4b < 0$, although $\lim\limits_{k\to\infty} g_k$ does not exist, it may reemerge in the form of a string of numbers (a cycle) the length of which depends on a specific relationship between *a* and *b* that, in turn, is determined by the roots of the polynomials $P_n(x)$. For example, when $a = 2$ and $b = -4$ it follows from (2) that $g_1 = 1$, $g_2 = -2$, $g_3 = 4$ and, once again, $g_4 = 1$. At the same time, $\frac{a^2}{b} = \frac{2^2}{-4} = -1$ and $P_1(-1) = 0$.

This note is a continuation of an earlier work by the authors [1, 2] in which it was demonstrated how the joint use of an electronic spreadsheet, *Maple* [7], and computer application *The Graphing Calculator* (*GC*) developed by Pacific Tech [4] enabled the discovery of cycles of integer periods formed by the solutions of equation (2). In what follows, the authors will demonstrate that further use of technology, including *Maple*, the *GC*, and (available free on-line) computational engine *Wolfram Alpha*, makes it possible to uncover several intriguing patterns in the very behavior of the cycles.

3. Parabolas as carriers of parameters yielding cycles

We say that the iterations g_k generated by equation (2) form a cycle of period *p* if $g_{k+p} = g_k \quad \forall k \in N$, where N is the set of natural numbers. For example, when $p = 3$ it follows from (2) that

$$g_{k+3} = a + \frac{b}{g_{k+2}} = a + \frac{b}{a + \frac{b}{g_{k+1}}} = a + \frac{b}{a + \frac{b}{a + \frac{b}{g_k}}}$$

or, setting $g_{k+3} = g_k = z$, $z = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{z}}}$. The last equality is equivalent to $z = a + \frac{abz + b^2}{(a^2 + b)z + ab}$

whence

$$(a^2 + b)z^2 + abz = a(a^2 + b)z + a^2 b + abz + b^2$$

or

$$(a^2 + b)(z^2 - az - b) = 0.$$

Because the assumption $a^2 + 4b < 0$ implies $z^2 - az - b \neq 0$, the equation

$$a^2 + b = 0 \tag{3}$$

serves as a condition that $g_{k+3} = g_k \quad \forall k \in N$. Furthermore, (3) implies $a^2 + 4b < 0$; that is, in the plane (a,b) the parabola $a^2 + b = 0$ resides inside the parabola $a^2 + 4b = 0$ and is a carrier of parameters that yield 3-cycles.

In turn, dividing both sides of (3) by b and setting $x = \frac{a^2}{b}$ we have $P_1(x) = x + 1 = 0$; that is, on the number line, the root of the polynomial $P_1(x)$ is responsible for the formation of a cycle of period three in equation (2). In much the same way, it can be shown that whereas $P_2(x) = x + 2$, the equation $z = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{z}}}}$ (in which $z = g_{k+4} = g_k$) is equivalent to $(a^2 + 2b)(z^2 - az - b) = 0$ or $a^2 + 2b = 0$ whence $P_2(x) = x + 2 = 0$ where $x = \frac{a^2}{b}$. That is, the root of the polynomial $P_2(x)$ is responsible for the formation of a cycle of period four in equation (2). Alternatively, the parabola $a^2 + 2b = 0$ resides inside the parabola $a^2 + 4b = 0$ and is a carrier of parameters that yield 4-cycles. In general, on the number line, a root of the Fibonacci-like polynomial $P_n(x)$ is responsible for the formation of a cycle of period *n* + 2 in equation (2) and in the plane (*a*, *b*) there exist $\lceil n/2 \rceil$ parabolas (where $\lceil x \rceil$ is the smallest integer greater than or equal to *x*) each of which is a carrier of parameters that yield those cycles.

Another interesting property of Fibonacci-like polynomials is that *all* their roots are real numbers and belong to the interval (-4, 0). While the latter part of this property (being equivalent to the fact that if $P_n(x^*) = 0$, then the parabola $a^2 - x^* b = 0$ resides inside the parabola $a^2 + 4b = 0$) can be proved without much difficulty for all $n = 0,1,2,...$, the former part, as mentioned elsewhere [1], can only be confirmed computationally for sufficiently large values of *n* by using, for example, *Maple*. Put another way, the statement that Fibonacci-like polynomials $P_n(x)$ defined through relations (1) don't have complex roots (alternatively, there exist $\lceil n/2 \rceil$ parabolas, all residing within the parabola $a^2 + 4b = 0$, each of which is a carrier of parameters that yield cycles of period $n + 2$) remains a computer-motivated conjecture. As Borwein put it, “there is no particular reason to think that all elegant true conjectures have accessible proofs” [5, p. 102].

The smallest root of a Fibonacci-like polynomial can be singled out for its unique influence on the behavior of cycles of any integer length generated by equation (2). When $\mathrm{mod}(p,2)\neq 0$, a different behavioral pattern of *p*-cycles associated with the largest root of a Fibonacci-like polynomial can be observed. The cycles, by representing a string of numbers, can be referred to as generalized golden ratios with the parabolas as their carriers. In what follows, the patterns exhibited by these ratios will be formulated in combinatorial terms and represented visually through circular diagrams. The role of technology in developing this apparently new knowledge about the generalized golden ratios will be critical. Indeed, several properties of the roots of Fibonacci-like polynomials utilized below will result from computing the effectiveness of which, however, requires intuition.

4. Permutations with rises and directions of cycles

Let $\hat{x}_{p,s}\in(-4,0)$ be the smallest root of the Fibonacci-like polynomial $P_{p-2}(x)$. Setting $b=\dfrac{a^2}{\hat{x}_{p,s}}$ in equation (2) yields

$$g_{k+1}=a+\frac{a^2}{\hat{x}_{p,\,s}\cdot g_k} \tag{4}$$

thus making the orbits g_k dependent on a single parameter. Furthermore, the inclusion $\hat{x}_{p,s}\in(-4,0)$ implies $b<0$ whence $a^2+4b<0$; conversely, the last inequality implies the inclusion $\dfrac{a^2}{b}\in(-4,0)$. As $P_{p-2}(\hat{x}_{p,s})=0$, a cycle of period *p* realizes on the parabola $b=\dfrac{a^2}{\hat{x}_{p,s}}$.

Consider the set

$$\{g_1(a), g_2(a), g_3(a), ..., g_p(a)\} \tag{5}$$

which represents a cycle of integer period *p* formed by the solutions of equation (4) where $g_1(a)=g_{p+1}(a)=1$. In the experimental mathematics fashion [6], the use of technology can motivate the following question: Is there any relationship among the elements of the above set that remains invariant as *p* changes?

In answering this question, the concept of permutation with rises [8] comes into play. It is said that the permutation $[r(1), r(2), r(3), ..., r(p)]$ has exactly *n* rises on the set Z_p if there exist exactly $n-1$ values of *i* such that $r(i)<r(i+1)$. For example, the permutations [1, 2, 3, 4, 5] and [1, 2, 3, 5, 4] have, respectively, five and four rises on Z_5. Indeed, in the former permutation $1<2<3<4<5$—a chain with four “<” signs and in the latter permutation $1<2<3<5>4$—a chain with three “<” signs.

We will say that the permutation $[r(1), r(2), r(3), ..., r(p)]$ determines the direction of cycle (5) on a certain interval I if for all $a\in I$ the inequalities

$$g_{r(1)}(a)>g_{r(2)}(a)>g_{r(3)}(a)>...>g_{r(p)}(a)$$

hold true. A computational experiment based on the use of the *GC* will result in the discovery of a pattern in the directions of a cycle on the number line that is invariant across the cycles of different lengths (periods). It will be shown that such a pattern can be formulated in terms of permutations with rises and is associated with the smallest root of a Fibonacci-like polynomial only. The case of the largest root will be considered separately.

5. Recognizing the nature of permutations of the elements of a three-cycle

Consider the polynomial $P_1(x) = x+1$ for which $P_1(-1) = 0$ and thus $\hat{x}_{3,s} = -1$. One can see that on the parabola $b = -a^2$ the three-cycle of the form

$$\{g_1(a), g_2(a), g_3(a)\} = \{1, a-a^2, \frac{a^2}{a-1}\} \quad (6)$$

realizes. Indeed, $g_1 = 1, g_2 = a + \frac{b}{g_1} = a - a^2, g_3 = a + \frac{b}{g_2} = a - \frac{a^2}{a-a^2} = \frac{a^2}{a-1}$ and

$g_4 = a + \frac{b}{g_3} = a - a^2 \div \frac{a^2}{a-1} = 1$. Furthermore, as shown in Figure 3, when $a < 0$ the inequalities $1 > \frac{a^2}{a-1} > a - a^2$ hold. The graphic demonstration can be confirmed analytically by reducing each inequality to an equivalent (true) inequality, $a^2 - a + 1 > 0$, as follows: $1 > \frac{a^2}{a-1} \Leftrightarrow a - 1 < a^2 \Leftrightarrow a^2 - a + 1 > 0$ and

$$\frac{a}{a-1} < 1 - a \Leftrightarrow a > (1-a)(a-1) \Leftrightarrow a > 2a - a^2 - 1 \Leftrightarrow a^2 - a + 1 > 0.$$

That is, $g_1(a) > g_3(a) > g_2(a)$. This implies that the permutation [1, 3, 2] determines the direction of cycle (6) on $(-\infty, 0)$. The graphs also show that the permutation [1, 2, 3] determines the direction of cycle (6) on $(0,1)$; that is, $1 > a - a^2 > \frac{a^2}{a-1}$. Indeed, $1 > a - a^2 \Leftrightarrow a^2 - a + 1 > 0$ and

$a - a^2 > \frac{a^2}{a-1} \Leftrightarrow 1 - a > \frac{a}{a-1} \Leftrightarrow (1-a)(a-1) < a \Leftrightarrow a^2 - a + 1 > 0$. Likewise, the permutation [3, 1, 2] determines the direction of cycle (6) on $(1, \infty)$, that is,

$\frac{a^2}{a-1} > 1 > a - a^2$. One can note that the permutations $[1, 3, 2]$, $[1, 2, 3]$ and $[3, 1, 2]$ have two, three, and two rises, respectively. Furthermore, the 3-cycle changes its direction twice. Finally, the three permutations can be represented through a circular diagram (Figure 4). The significance of this representation will be revealed later as we consider strings of oscillating golden ratios of higher lengths.

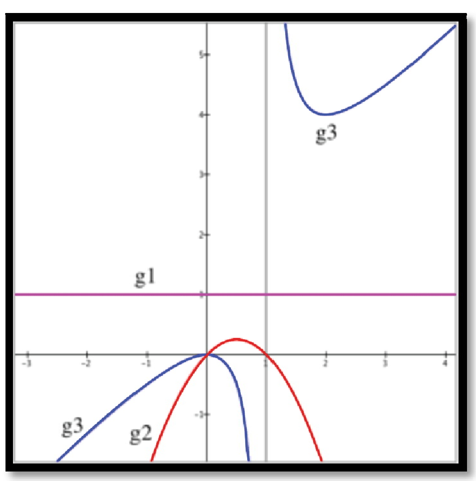

Figure 3. Graphs of the elements of cycle (6).

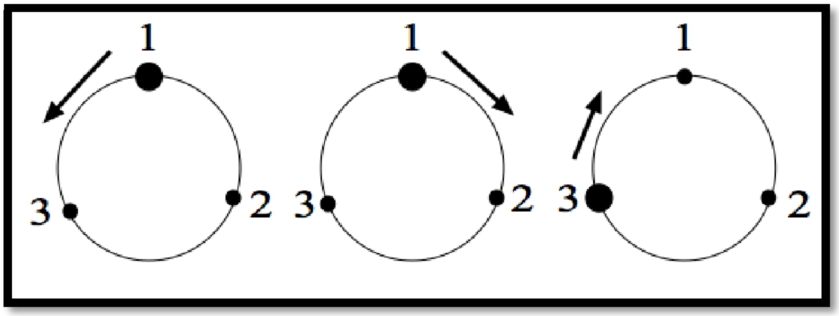

Figure 4. Circular diagram for cycle (6).

6. Permutations of the elements of a four-cycle

Consider the polynomial $P_2(x)$. According to definition (1), $P_2(x) = x^{\text{mod}(2,2)} P_1(x) + P_0(x) = x + 1 + 1 = x + 2$ whence $P_2(-2) = 0$. Consequently, on the parabola $b = -\frac{a^2}{2}$ a cycle of period four realizes. This cycle,

$$\{g_1(a), g_2(a), g_3(a), g_4(a)\} = \{1, \frac{a(2-a)}{2}, \frac{a(a-1)}{a-2}, \frac{a^2}{2(a-1)}\}, \tag{7}$$

can be calculated with the help of *Maple* through the following code: $g1 := 1;\, b := -\frac{a^2}{2};$ $g2 := a + b;\, g3 := simplify(a + \frac{b}{g2});\, g4 := simplify(a + \frac{b}{g3})$. One can conclude the calculations by verifying that $g5 := simplify(a + \frac{b}{g4})$ yields the value of $g1$.

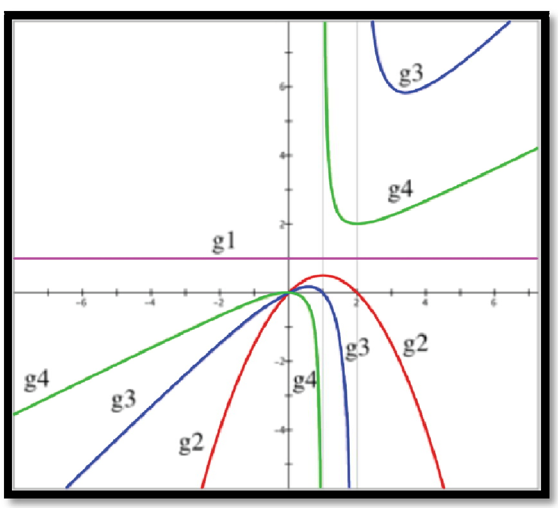

Figure 5. Graphs of the elements of cycle (7).

As shown in Figure 5, for $a < 0$ the inequalities $g_1(a) > g_4(a) > g_3(a) > g_2(a)$ hold true. The graphic demonstration can be confirmed analytically by employing the *Wolfram Alpha* to carry out symbolic calculations (Figure 6). That is, for all $a<0$ the inequalities $1>\frac{a^2}{2(a-1)}>\frac{a(a-1)}{a-2}>\frac{a(2-a)}{2}$ hold true.

Therefore, the permutation [1, 4, 3, 2] determines the direction of cycle (7) for $a<0$. This time, the four-cycle changes the direction three times as it passes through the points a = 0, a = 1, and a = 2 all of which are zeros of the elements of cycle (7) and through which (except a = 0) their vertical asymptotes pass.

Observing the graph in Figure 5 makes it possible to conclude that the permutations [1, 2, 3, 4], [4, 1, 2, 3], and [3, 4, 1, 2] determine, respectively, the directions of cycle (7) in the intervals (0, 1), (1, 2), and $(2,\infty)$. The permutations $[1,4,3,2]$, $[1,2,3,4]$, $[4,1,2,3]$ and $[3,4,1,2]$ have two, four, three, and three rises, respectively. Furthermore, the 4-cycle changes its direction three times. One can note that in each of the permutations the neighboring elements are also consecutive points on the circles (Figure 7) and follow each other either in the CW or CCW direction, a property that may be overlooked without using circular diagrams as alternative representations of permutations. This fact one more time points at the importance of multiple representations of a mathematical concept using different notation systems [11]. Furthermore, as will be shown below, this kind of oscillations of generalized golden ratios observed in the context of the circular diagram notation, only relates to the case of cycles formed by the smallest root of Fibonacci-like polynomials.

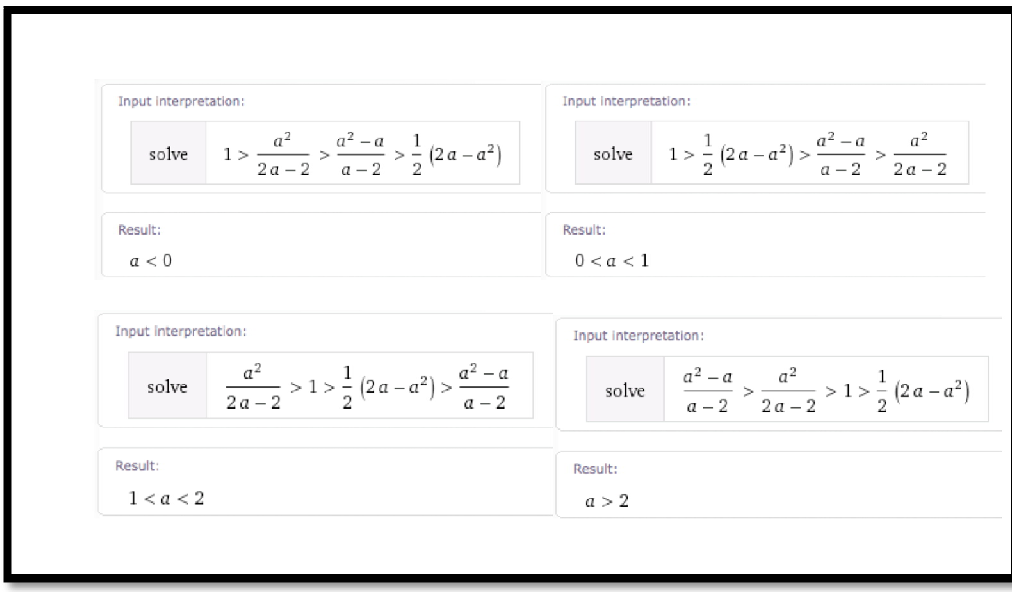

Figure 6. Confirming graphic demonstration.

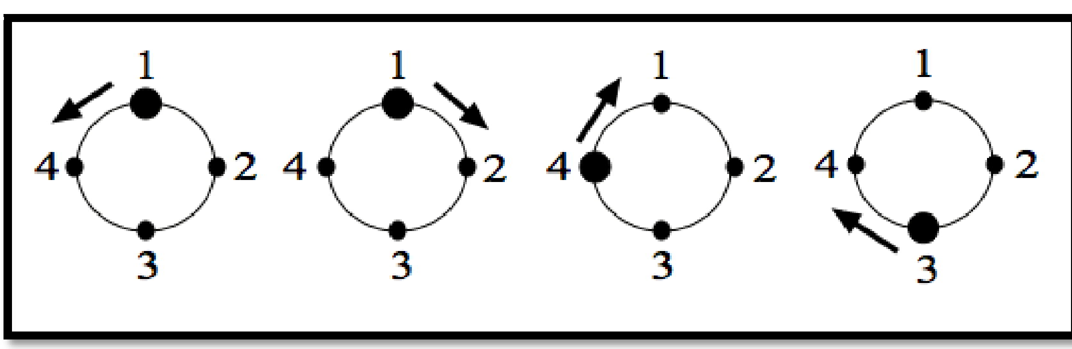

Figure 7. Circular diagram for cycle (7).

7. Permutations of the elements of a five-cycle

So far, our explorations concerned Fibonacci-like polynomials with a single root. It follows from definition (1) that the first polynomial with two roots is

$$P_3(x) = x^{\mathrm{mod}(3,2)} P_2(x) + P_1(x)$$
$$= x[x^{\mathrm{mod}(2,2)} P_1(x) + P_0(x)] + P_1(x) = x(x+1+1) + x + 1 = x^2 + 3x + 1.$$

The smallest root of the equation $x^2 + 3x + 1 = 0$ is equal to $-(3+\sqrt{5})/2 \in (-4,0)$. Using *Maple*, one can develop the following cycle generated by this root:

$$\{g_1(a), g_2(a), g_3(a), g_4(a), g_5(a)\}$$
$$= \{1, \frac{a(3+\sqrt{5}-2a)}{3+\sqrt{5}}, \frac{a(1+\sqrt{5}-2a)}{3+\sqrt{5}-2a}, \frac{-2a(-1-\sqrt{5}+a+\sqrt{5}a)}{(3+\sqrt{5})(1+\sqrt{5}-2a)}, \frac{(-1+\sqrt{5})a^2}{(-1-\sqrt{5}+a+\sqrt{5}a)}\} \quad (8)$$

and check to see that $g(6) = a + \dfrac{2(-1-\sqrt{5}+a+\sqrt{5}a)}{(-3-\sqrt{5})(-1+\sqrt{5})} = 1.$

Using the *GC*, one can construct the graphs of $g_i(a), 1 \le i \le 5$, in a single drawing and then, prompted by the graphical representation, to verify symbolically the following inequalities: $g_1(a) > g_5(a) > g_4(a) > g_3(a) > g_2(a)$ for $a < 0$, $g_1(a) > g_2(a) > g_3(a) > g_4(a) > g_5(a)$ for $0 < a < 1$, $g_5(a) > g_1(a) > g_2(a) > g_3(a) > g_4(a)$ for $1 < a < (1+\sqrt{5})/2$, $g_4(a) > g_5(a) > g_1(a) > g_2(a) > g_3(a)$ for $(1+\sqrt{5})/2 < a < (3+\sqrt{5})/2$, and $g_3(a) > g_4(a) > g_5(a) > g_1(a) > g_2(a)$ for $a > (3+\sqrt{5})/2$. As the expressions for $g_i(a), 1 \le i \le 5$, become more and more complicated in comparison with cycle (7), one can use *Maple* in verifying the first chain of the above inequalities through the code $solve(\{g1 > g5, g5 > g4, g4 > g3, g3 > g2\}, a)$, where the elements of the four inequalities are notations used in *Maple* to define and calculate those expressions. Similar code can be used to verify the other four chains of simultaneous inequalities.

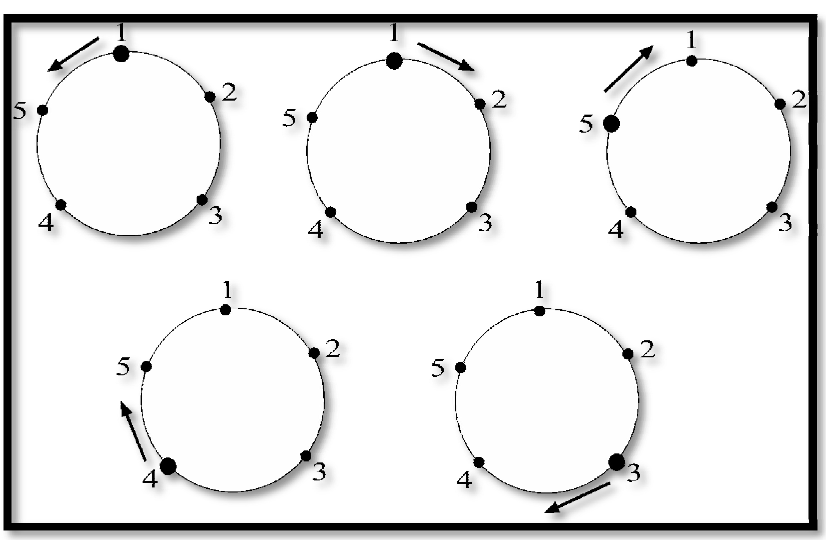

Figure 8. Circular diagram for cycle (8).

Therefore the permutations

[1, 5, 4, 3, 2], [1, 2, 3, 4, 5], [5, 1, 2, 3, 4], [4, 5, 1, 2, 3], and [3, 4, 5, 1, 2]

characterize directions of cycle (8) in the intervals $(-\infty,0),(0,1),(1,(1+\sqrt{5})/2),((1+\sqrt{5})/2,(3+\sqrt{5})/2)$, and $((3+\sqrt{5})/2,\infty)$, respectively. These five permutations have two, five, four, four, and four rises, respectively. Furthermore, the 5-cycle changes its direction four times. The circular diagram representing the permutations is shown in Figure 8. Once again, the neighboring elements of the permutations follow each other on the circles as well, either in the CW or CCW direction.

8. Generalizing from observations

As one analyzes the above three sets of permutations on Z_3, Z_4, and Z_5 the following pattern emerges. In all the three cases, regardless of the period, there are always one permutation with two rises and one permutation with the number of rises equal to the period of the corresponding cycle. Furthermore, given the period, other permutations have the same number of rises: for period three—two rises, for period four—three rises, and for period five—four rises; in other words, the number of rises in the remaining permutations is one smaller then the corresponding period. Analyzing the circular diagrams of Figure 4, Figure 7 and Figure 8, the observed pattern can be formulated as follows. In all the three cases, there are two circles where starting from point 1 and passing through all consecutive points the moves both in CW and CCW directions can be observed. On the rest of the circles the like moves in the CW direction starting from all the points but 2 occurs.

In general, there is only one permutation with two rises that starts with one; namely, $[1,p,p-1,p-2,...,2]$ — the CCW movement on a circle with *p* points. There is only one permutation with *p* rises; namely, $[1,2,3,...,p-1,p]$— the CW movement on this circle. The remaining *p* – 2 permutations can be subsequently generated from the one with *p* rises, by putting the last element on the first place; namely $[p,1,2,3,...,p-1]\to[p-1,p,1,2,...,p-2]\to[p-2,p-1,p,1,...,p-3]\to[3,4,...,p,1,2]$— all these correspond to movements in the CW direction on the circles avoiding the start from the point 2. A formal proof of the above technology-motivated generalization can be found elsewhere [2].

9. Can a pattern be found for oscillations formed by other roots?

The results of this section are based on computational experiments informed by guesstimate. First, it follows from definition (1) that among any three consecutive Fibonacci-like polynomials, there are exactly two polynomials of the same degree. In addition, computing shows that three consecutive polynomials don't have roots in common. However, because a cycle of length *p* can be construed as a cycle the length of which is a multiple of *p*, the corresponding Fibonacci-like polynomials do have roots in common. Using *Maple*, one can discover that when *p* is an odd number the largest root of the corresponding Fibonacci-like polynomial that generates a *p*-cycle coincides with the root which generates a $(2p)$–cycle. The further period doubling investigations lead to polynomials the largest root of which is different from that of their immediate predecessor. For example, the largest root of $P_3(x)$ (forming a 5-cycle) coincides with that of $P_8(x)$ (thus forming a 10-cycle from two identical 5-cycles) and is different from that of $P_{18}(x)$(forming a 20-cycle). Therefore, for the sake of simplicity, only cycles of odd length formed by the largest root of the corresponding Fibonacci-like polynomial will be considered below.

The first Fibonacci-like polynomial having more than one root and associated with odd length is $P_3(x)=x^2+3x+1$. Once again, the use of *Maple* makes it possible to develop the following cycle generated by $\hat{x}_{5,l} = (-3+\sqrt{5})/2$, the largest root of $P_3(x)$,

$$\begin{aligned}&\{g_1(a), g_2(a), g_3(a), g_4(a), g_5(a)\}\\ &= \{1, \frac{a(-3+\sqrt{5}+2a)}{-3+\sqrt{5}}, \frac{a(-1+\sqrt{5}+2a)}{-3+\sqrt{5}+2a},\\ &\frac{2a(1-\sqrt{5}-a+\sqrt{5}a)}{(-3+\sqrt{5})(-1+\sqrt{5}+2a)}, \frac{(1+\sqrt{5})a^2}{(1-\sqrt{5}-a+\sqrt{5}a)}\}.\end{aligned} \tag{9}$$

Next, just like in the case of other cycles, the graphs of the elements $g_i(a)$, $1 \le i \le 5$, can be constructed by using the *GC*. These graphs can assist one in developing permutations through solving five chains of four inequalities using *Maple* that indicate the type of oscillations of the corresponding generalized golden ratios. As a result, the permutations

$$[4, 1, 3, 5, 2], [1, 3, 5, 2, 4], [1, 4, 2, 5, 3], [3, 1, 4, 2, 5], [5, 3, 1, 4, 2] \tag{10}$$

can be shown to characterize the directions of cycle (9) within the intervals

$$(-\infty, -(\sqrt{5}-1)/2), (-(\sqrt{5}-1)/2, 0), (0, (3-\sqrt{5})/2), ((3-\sqrt{5})/2, 1), (1, \infty).$$

In combinatorial terms, among permutations (10), there are one permutation with two rises, one permutation with four rises, and three permutations with three rises. The pattern observed earlier for the smallest root of a Fibonacci-like polynomial where, for example, a permutation with five rises was found, is not confirmed now. In turn, one can construct the circular diagram (Figure 9) which shows a different type of behavior: the neighboring elements of the permutations are not consecutive ones on the circles. Rather, in all the five cases the neighboring elements of the permutations are separated by exactly one element on a circle. Also, there are two permutations that start with 1 moving in the CW and CCW directions and no permutation starting with 2. Finally, the first permutation starts with 4. Can a similar behavior be observed in the case of a cycle of period seven associated with the largest root of the polynomial $P_5(x)$?

It follows from definition (1) that $P_5(x) = x^3 + 5x^2 + 6x + 1$, a polynomial with three real roots. Considering the cycle

$$\{g_1(a), g_2(a), g_3(a), g_4(a), g_5(a), g_6(a), g_7(a)\} \tag{11}$$

generated by its largest root, $\hat{x}_{7,l} \cong -0.198062264195$, one can conclude (using *Maple*) that the permutations

$$\begin{aligned}&[4, 6, 1, 3, 5, 7, 2], [6, 1, 3, 5, 7, 2, 4], [1, 3, 5, 7, 2, 4, 6], [1, 6, 4, 2, 7, 5, 3],\\ &[3, 1, 6, 4, 2, 7, 5], [5, 3, 1, 6, 4, 2, 7], [7, 5, 3, 1, 6, 4, 2]\end{aligned} \tag{12}$$

define the directions of cycle (11) on the number line.

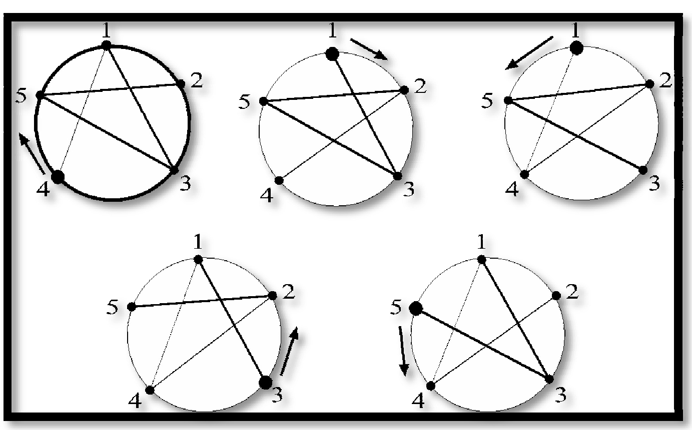

Figure 9. Circular diagram for cycle (9).

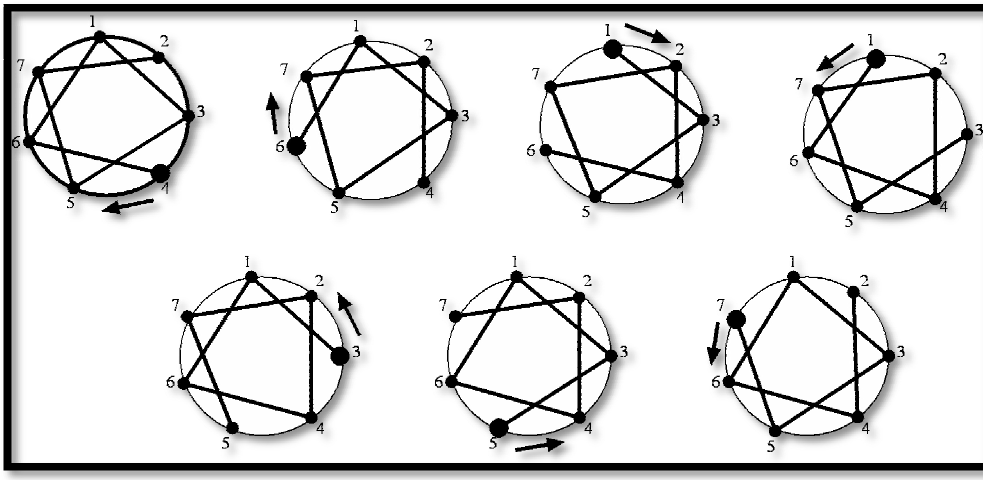

Figure 10. Circular diagram for cycle (11).

Among permutations (12), there are one permutation with six rises, one permutation with two rises, two permutations with five rises, and three permutations with three rises. Whereas in terms of the rises, no pattern that matches the case of cycle (9) emerges, by constructing the circle diagram shown in Figure 10, one can see that in all the seven cases the neighboring elements of the permutations are separated by exactly one element on a circle. Furthermore, there are two permutations that start with 1 moving in the CW and CCW directions and no permutation starting with 2. Finally, the first permutation starts with 4. But this is exactly the type of oscillations shown in Figure 9 for the (five-element) string of generalized golden ratios also formed by the largest root of the polynomial $P_3(x)$.

Continuing in this vein, for a 9-cycle generated by the largest root of the polynomial $P_7(x)$, one can find that the permutations

[4, 6, 8, 1, 3, 5, 7, 9, 2], [6, 8, 1, 3, 5, 7, 9, 2, 4], [8, 1, 3, 5, 7, 9, 2, 4, 6],
[1, 3, 5, 7, 9, 2, 4, 6, 8], [1, 8, 6, 4, 2, 9, 7, 5, 3], [3, 1, 8, 6, 4, 2, 9, 7, 5],
[5, 3, 1, 8, 6, 4, 2, 9, 7], [7, 5, 3, 1, 8, 6, 4, 2, 9], [9, 7, 5, 3, 1, 8, 6, 4, 2], (13)

define the cycle's directions on the number line. Once again, by representing permutations (13) through a circular diagram (not shown here), one can observe that the first permutation starts with 4; two permutations start with 1 and they move in the CV and CCW directions; no permutation starts with 2; and the neighboring elements of the permutations are separated by exactly one element. One can guess that similar behavior could be observed for odd $p > 9$, as on a circle, augmenting a set of points by two points enable for that kind of oscillations to continue.

10. Concluding remarks

This note demonstrated how, using commonly available computer applications, allows for the discovery of patterns that structure oscillations of generalized golden ratios formed by the roots of the Fibonacci-like polynomials. It was shown how universality of oscillations associated with the smallest root, regardless of the length of the string of the ratios, can be established and formulated in combinatorial terms. In addition, the use of circular diagrams informed by computational experiments in the case of the strings of odd number lengths made it possible to recognize patterns associated with the largest root. The note may be used as a source of activities for secondary mathematics teacher candidates enrolled in a capstone course emphasizing the value of learning mathematics through a computational experiment.

Regarding the relationship between the use of technology and the development of skills in algebra by a student of mathematics, the authors are in the opinion that simple symbolic calculations, like in the case of a three-cycle, must be encouraged to make sure that the basic algebraic skills have been developed and can be used in analyzing results produced by a computer algebra system. Yet, as the complexity of symbolic calculations increases, the use of technology should also be encouraged and promoted as appropriate. Through such a use one develops a new set of skills and, more importantly, extends the boundaries of symbolic calculations that otherwise would either be impossible or require time that can be used for other, more meaningful and useful mathematical activities.

Thus, as far as mathematics learning and knowledge development are concerned, the appropriate use of technology can be interpreted in the following two complimentary ways: what can be done with technology today can be done without technology tomorrow (paraphrasing the viewpoint, "What the child is able to do in collaboration today, he will be able to do independently tomorrow" [12, p. 220]) and what yesterday could be done without technology, today can be extended to include the use of technology (paraphrasing the viewpoint, "numerical experimentation . . . can help us decide what to believe in mathematics" [5, p. 104]).

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