EXPERIMENTAL PRE-COLLEGE MATHEMATICS: THEORY, PEDAGOGY, TOOLS

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Abstract

The paper introduces the computational experiment approach to school mathematics curriculum by investigation a variety of mathematical models that were typically considered advanced in the pre-computer age. This approach makes it possible to connect sophisticated mathematical context and the modern day teaching practice. The talk will demonstrate how mathematical experimentation in the technological paradigm creates conditions for collateral learning to occur including the development of skills important for engineering applications of mathematics.

Key words: experiment, technology, signature pedagogy, collateral learning, teacher education.

ZDM Subject Classification: **I70, N70, R20**

1. Introduction

In this paper, the word experiment stems from the use of computing technologies in support of pre-college mathematics curriculum, in particular, as it is introduced by the author in mathematics teacher education courses. These modern digital tools when integrated with mathematics instruction have great potential to create and enhance conditions for learners' inquiry into mathematical structures represented through interactive graphs, dynamic geometric shapes, and electronically generated and controlled arrays of numbers. In the context of mathematical education in general, experimental mathematics makes use of such representations as a way of motivating learners' conjecturing of mathematical propositions followed up (whenever possible) by their formal demonstration.

In developing ideas about experimental pre-college mathematics, a number of commonly available computer applications allowing for lucid presentation of grade-appropriate mathematical ideas can be used. One such application is an electronic spreadsheet used to support a variety of numeric calculations. The tool was conceptualized in educational terms by its inventor, Dan Bricklin, as 'an electronic blackboard and electronic chalk in a classroom' [40]. Another application is the *Graphing Calculator 4.0* produced by Pacific Tech [8] that facilitates experimentation in algebra through the software's capability of constructing graphs from any two-variable equation, inequality, or a combination of those. A remarkable computer program available free on-line is *Wolfram Alpha* developed by

Wolfram Research – a tool that allows for different types of experimentation with mathematical concepts, including the construction of graphs of functions and relations and carrying out complicated symbolic computations. Mathematical experiments can also be carried out in the context of *The Geometer's Sketchpad*, a dynamic geometry application created by Nicholas Jackiw in the late 1980s. Nowadays, especially in Europe, GeoGebra [22] – a free, open-source application for teaching and learning geometry and algebra – is used extensively making it another appropriate tool for mathematical experimentation. Finally, briefly in this paper some computational results carried out by *Maple* [11] – a powerful tool for mathematical modeling – are presented.

Whereas the notion of experiment in the context of education has multiple meanings, learning as the goal of experiment is what all the meanings have in common. In a seminal book on experiment in education, McCall [31] recognized the power of experiment as a milieu where "teachers join their pupils in becoming question askers" (p. 3). Mathematics is especially conducive to the development of an environment in which reflective inquiry, referred to by Dewey [16] as a problem-solving method that blurs the distinction between knowing and doing by integrating knowledge with experience, is the major learning strategy.

The bedrock of any mathematical experiment is observation – an activity of the mind that feeds reflective inquiry and builds up experience. Euler, the father of modern mathematics, including number theory, emphasized the important role of observations and so-called quasi-experiments or thought processes (experiments) that stem from observations: "the properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstration" (Euler, cited in [39, p. 3]). Yet, as Euler advised, "we should take great care not to accept as true such properties of the numbers which we have discovered by observation and ... should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful" (ibid, p. 3). In that, Euler pointed at the importance of formal justification of the results of a mathematical experiment. These results, by providing basis for insight, give birth to theory. In reciprocity, theory can be used to support an experiment as its conditions become more and more complex. In this paper, several examples of interplay between theory and the grade appropriate modern day mathematical experiment will be provided.

2. Developing interest in mathematics through a computational experiment

At the end of the 19th century, an American psychologist William James studied the art of teaching (called pedagogy) from the point of view of psychology. One of his ideas about developing learners' interest in a subject matter was as follows: "Any object not interesting in itself may become interesting through becoming associated with an object in which an interest already exists" [25, p. 62]. More than a century later, the National Council of Teachers of Mathematics [34] – the major professional organization of mathematics educators in North America – asserted, "Effective teachers optimize the potential of technology to develop students' understanding, stimulate their *interest*, and increase their proficiency in mathematics" (p. 1, italics added). What James had proposed long before the digital era can be applied to the modern technology-enhanced classroom. The experimental mathematics approach can provide interplay between computing (experiment) and formal demonstration (theory). This approach can utilize technology for the development of reasoning skills in the context of pre-college mathematics curricula of teacher education programs.

3. Defining computational experiment in the pre-college context

3.1. International dimension of the use of technology

The notion of computational experiment as an inquiry into mathematics brings about the term experimental mathematics in the modern pre-college classroom. The advent of computers made it possible to extend the notion of experimental mathematics to include the idea of advanced problem solving made possible by computationally supported mathematical education environments. Through such inquiry, both the complexity of mathematics to be involved and the sophistication of technology to be used can vary on the spectrum from precollege level to that of mathematical research. Although the use of paper and pencil allows for some basic computations also, the current emphasis on the use of technology in the teaching of mathematics in the countries (referring to sources available in English) like Australia [36], Canada [19, 37], England [7, 15], Japan [46], Singapore [33], the United States [12, 13, 23, 24, 34, 35, 41], and other places in the world as a way of making its learning more accessible is the main reason for the author to emphasize the use of the tools available in the digital era. So, in the context of this paper, the term experimental mathematics means an approach to mathematics teaching and learning made possible by the use of various commonly available and user friendly computational tools. Furthermore, methodology of the experiment remains the same regardless at which grade level it is used and what tools it employs.

3.2. Bridging the gap between the past and the present

Introducing experimental mathematics approach into pre-college mathematics curriculum, note that, unlike experimentation in mathematics research, the approach, usually, does not offer results that were not possible to obtain in the pre-digital era. For example, through unsophisticated experimentation with the first two terms of Fibonacci sequence F_n defined recursively as

$$F_{n+2} = F_{n+1} + F_n , n = 0, 1, 2, ..., F_0 = F_1 = 1 , \qquad (1)$$

one can "discover" that regardless of the values F_0 and F_1, the ratios F_{n+1} / F_n always tend to the same number known as the Golden Ratio. What is hidden in such a computational experiment is that the *n*-th term of the Fibonacci-like sequence $x, y, x+y, x+2y, 2x+3y, ...,$ in which each term beginning from the third is the sum of the previous two terms, has the form $f_n(x,y) = F_{n-2}x + F_{n-1}y$ and that for all (real) x and y $\lim_{n \to \infty} \dfrac{f_{n+1}(x,y)}{f_n(x,y)} = \dfrac{1+\sqrt{5}}{2}$ — the Golden Ratio. This fact was known long before the digital era and it can be confirmed through formal demonstration without much difficulty. What was not known until recently [6] is that if the difference equation in (1) is parameterized to the form $F_{n+2} = aF_{n+1} + bF_n$, where $a, b \in R$, the $\lim_{n \to \infty} F_{n+1} / F_n$ may cease to exist and, instead, the ratios may be attracted by the strings of numbers of any given length. In other words, for some values of *a* and *b* the ratios may form cycles of any given period.

However, in general, what a computer allows mathematics educators to achieve is to bridge the gap between pedagogical practices of the past and the present. Mathematical content such as Fibonacci numbers and the Golden Ratio available in the past to only very few mathematically advanced students, presently can be accessed by the majority of students through the use of technology. Fibonacci numbers and the Golden Ratio represent an

example of such content. Another example of how a computer can bridge this gap between the past and the present is the availability of software that can graph both functions and relations.

To clarify, consider the equation $x^2 + ax + 1 = 0$ with variable x and parameter a. By constructing its locus using *Wolfram Alpha* (or the *Graphing Calculator 4.0*), one can immediately see (Figure 1) that when $|a| < 2$ this equation does not have real roots because no line $a = const$ for those values of a has a point in common with the locus. To obtain this result without a computer would require one first to convert the quadratic equation to the form $a = -x - \dfrac{1}{x}$ and then to construct the graph of the function $f(x) = -x - \dfrac{1}{x}$. To construct the graph without technology, one has to use a number of facts from the secondary mathematics curriculum including the behavior of $f(x)$ in the neighborhood of zero and at infinity, the arithmetic mean-geometric mean inequality $x + \dfrac{1}{x} \geq 2$ for $x > 0$, and the symmetry of the graph of $f(x)$ about the origin. At the same time, a computational experiment requires one only to use skills in interpreting the graph constructed by software (provided, of course, that a user can correctly enter the quest for graphing into the input box of software used).
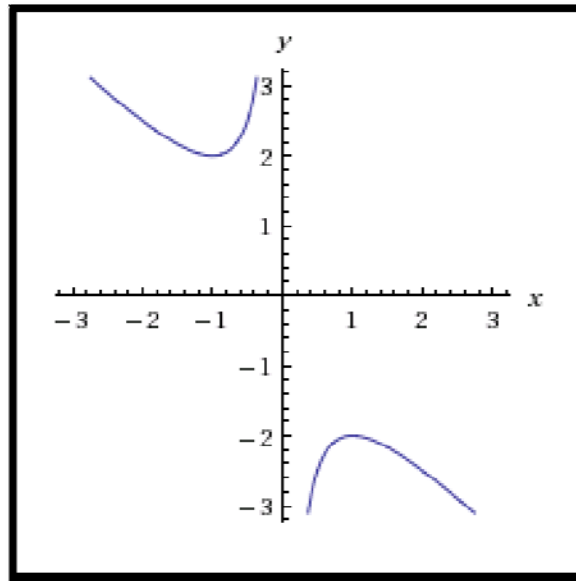

Figure 1. The locus of the equation $x^2 + ax + 1 = 0$.

4. Mathematical experiment as signature pedagogy

4.1. Three structures of signature pedagogy

Shulman [44] introduced the notion of signature pedagogy in the context of studies of medicine and law and argued for the importance of developing in students the habits of mind of professionals working in the field they are preparing to join. This notion was explored for a variety of disciplines [21], including mathematics [3, 18, 27, 38]. Common characteristics for all signature pedagogies varying across disciplines comprise three entities – surface structure of teaching, deep structure of teaching, and implicit structure of teaching – called by Shulman [44] the structures of signature pedagogy. When a teacher possesses only very basic

subject matter knowledge, their pedagogical skills and abilities lie within the surface structure of teaching. In mathematics, teaching at the surface structure level fails to appreciate such pillars of knowledge development as problem solving, reflective inquiry and conceptual learning. Instead, someone holding to the surface structure of mathematics teaching emphasizes memorization of rules without understanding their meaning. Mathematical experiment cannot be used in the classroom as a teaching method unless a teacher is prepared psychologically to gravitate away from the surface structure of teaching.

4.2. Deep structure of teaching mathematics

A pedagogy used to support the deep structure of teaching is based on one's good knowledge of content he or she teaches. This pedagogy requires one's understanding how different areas and concepts of a discipline are connected. It is open to the practice of teachers and students exploring jointly mutually generated questions. In mathematics, deep structure of teaching implies the need for a teacher to know mathematics, understand its main ideas and concepts, be able to demonstrate and interpret connections among the concepts, and have a rich repertoire of motivational techniques for the introduction of such concepts. In other words, a teacher must have a strong command of pedagogical content knowledge [42, 43], currently considered as the basis for students' progress in learning mathematics [9]. Motivational techniques may include computer experimentation with mathematical concepts as was suggested more than 40 years ago: "the computer provides mathematicians with an unparalleled opportunity to motivate students towards experimentation with mathematics" [32, p. 295]. Therefore, with the advent of technology, deep structure of mathematics pedagogy can also be characterized by a mathematical experiment that stems from one's understanding of how mathematics and technology interact.

4.3. Three descriptors of signature pedagogy

Teaching students to do mathematics is inherently linked to Shulmans' [45] three descriptors of signature pedagogy—uncertainty, engagement, and formation. If a teacher incorporates reflective inquiry pedagogy, thereby acting at the deep structure of teaching, a student, by asking an unexpected question, can pose a "new" problem. This problem might be too difficult to solve even in an experimental fashion. The realization of this fact points to the uncertainty of mathematics pedagogy. Next, doing something presupposes engagement; so problem-solving focus of current mathematics pedagogy does require students' engagement. Finally, regardless of the outcome of this engagement, one develops a kind of professional disposition towards the discipline of mathematics even when a problem is not solved from the first attempt. In this case, it is very important for a teacher to provide students with a qualified assistance. As noted by de Lange [14], the students' excitement with problem solving and the teacher's growing confidence with mathematical content have great potential to overcome the issue of uncertainty in the classroom. In the digital era, at any grade level, problem solving and mathematical exploration can be supported by the use of computers, an innovation which brings the elements of uncertainty, engagement, and formation to the mathematics classroom. Therefore, these ever changing teaching tools, in Shulman's [44] words, "create an opportunity for reexamining the fundamental signatures we have so long taken for granted" (p. 59).

5. Two types of technology application

Maddux [28] introduced the notion of Type I vs. Type II applications of technology referring to the latter type as "new and *better* ways of teaching" (p. 38, italics in the original).

In the context of pre-college mathematics, an emphasis on better ways of teaching is due to the need to move away from using a computer for drill and practice with its either-right-or-wrong educational philosophy or for transmitting (often formal) mathematics instruction in a mode which is inherently entertaining. The most common Type I applications of technology to mathematics education are those that support the above two pedagogical approaches – drill and practice and entertainment. On the contrary, solving multistep problems or exploring curricular topics that otherwise are not attainable are examples of Type II application of technology in mathematics education. In the problem-solving-oriented mathematics education, using technology for analysis, conjecturing and justification, when ideas are born in the minds of students who are assisted in this process by a teacher as a 'more knowledgeable other', does encourage individualized instruction as the core of the Type II concept [29].

The concept of Type I/Type II application of technology turned out to be a very powerful theoretical shield against sometimes rather strong critique and persistent skepticism regarding the worth and purpose of using computers in the schools. More recently, Maddux and Johnson [29] argued that "the boring and mundane uses to which computers were often being applied [at the infancy of their educational applications] had set the stage for a major backlash against bringing computers into schools" (p. 2). Therefore, the experimental mathematics approach cannot be viewed an educationally successful tool unless a teacher acts at the deep structure level of signature pedagogy, is capable of dealing with uncertainty when supporting students' engagement in computationally enabled problem solving, and has skills to foster mathematical mindset of the students.

Encouraging reflection and supporting analysis of the action by a student implies that one acts at the deep structure level of teaching. This kind of professional behavior requires broad pedagogical knowledge of what a specific computer environment affords, intellectual courage to motivate students to reflect on their actions, readiness to answer unexpected questions, and willingness to support students' natural curiosity that develops within the "zone of proximal development" [50]. When a student's performance becomes fully assisted, the teaching occurs at the deep structure level. Teaching at that level also requires knowledge of current national standards of the subject matter taught, understanding connections which exist between concepts that belong to different grade levels, and, more importantly, skills of using computers to support concept learning. When a computer is used at the deep structure level, Type II application of technology occurs.

6. Two styles of assistance in the digital era

Teachers' beliefs about mathematics pedagogy and instructional uses of computers are the major elements of their implicit structure of signature pedagogy. An individual teaching philosophy either keeps the teacher at the surface structure level or motivates a quick transition from one level to another. Experience working at the deep structure level, in turn, affects the extent of the richness of the implicit structure of the pedagogy.

Similarly to the two types of technology integration, one can talk about two styles of assistance that teachers can offer to their students [3]. Style I assistance is typified by the surface structure of teaching and it is limited by one's teaching philosophy which does not view teaching, for the most part, as assisted performance [48]. Style II assistance is typified by the deep structure of teaching and it is open to promoting reflective inquiry and taking an intellectual risk by going into an uncharted territory brought to light through an open-ended classroom discourse. Likewise, teachers' superficial knowledge of technology offers students Style I assistance only. By the same token, Style II assistance in the students' design and/or

utilization of a computational learning environment requires a high level of technological literacy on the part of a teacher. These two styles of assistance, observed within the general instructional setting, underlie one's implicit structure of signature pedagogy and determine the composition of two other structures.

7. Parallel structures of teaching and learning

7.1. Linking teaching to learning

The theoretical construct of signature pedagogy can be augmented to include students as the beneficiaries of the pedagogy. This augmentation is consistent with the underlying principles of educational scholarship, which sees the concept of signature pedagogy as an application of the theory of learning to the practice of teaching [44]. In particular, in the context of mathematical education this concept serves as a link between teaching and learning [18]. However, in general, the proposed extension is neither grade nor content specific and it may be applied to any discipline.

By extending the notion of signature pedagogy to include students, two separate but interdependent universes can be considered: teacher's universe and student's universe. Each universe comprises three levels echoing Shulman's classical structures of signature pedagogy, which can be then considered as a part of the whole teaching and learning process. In this process, teaching affects learning and vice versa; that is, the way students learn (or aspire to learn) can affect the way teachers teach. Due to such reciprocity of teaching and learning, the same three structures — surface structure of learning, deep structure of learning, and implicit structure of learning — can be considered in the student's universe.

7.2. A need for Style II assistance in the zone of proximal development

The teacher's and the student's universes consist of matching parts which constantly affect each other as both participants of the process of education navigate through the structures of teaching and learning. In the presence of computers, teaching at the surface structure keeps a student at the surface structure of learning when the use of a computer is expected only to support enjoyment or, at most visualization of some very basic mathematical structures such as the multiplication table (Figure 2). Yet, the teacher cannot always control students' use of technology. In the context of "playing" with numbers within a spreadsheet-based multiplication table, a student might recognize patterns that the program generates and then ask the teacher various questions about those patterns. For example, why does a path connecting by a k-gnomon two identical factors k (Figure 2) comprise the products the sum of which is equal to the third power of the factor, k^3? Or, why do the pairs of products equidistant from the borders of the multiplication table have equal sums? Of course, if the sums of products within a gnomon (like the one connecting the threes where $3 + 6 + 9 + 6 + 3 = 3^3$) are not recognized as cubed integers in a numeric environment, it is unlikely that one would be able to proceed from experiment to theory and to establish at the symbolic level within the k-gnomon the following chain of equalities

$$k + 2k + 3k + ... + (k-1)k + k^2 + (k-1)k + ... + 3k + 2k + k$$

$$= 2[k + 2k + 3k + ... + (k-1)k] + k^2 = 2k(1 + 2 + 3 + ... + k - 1) + k^2$$

$$= 2k \cdot \frac{(k-1)k}{2} + k^2 = k^2(k-1+1) = k^3,$$

from where, taking into account [35, p. 337] that the sum of all numbers in the multiplication table is equal to

$$(1 + 2 + 3 + \ldots + n)^2 = \left[\frac{n(n+1)}{2} \right]^2,$$

the formula $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2} \right]^2$ results.

One can ask: why do the numbers within a gnomon add up to a cubic number? This question calls for an explanation of the interplay between the procedural perspective on algebraic symbolism and the conceptual understanding of a symbol, k^3, involved. Through such an explanation one can appreciate a link that exists between thinking of symbol as a process and seeing it as a concept [47]. To this end, note that the right-bottom element of a k-gnomon, being a symbol representing a square number, can be interpreted as one of the k layers of a $k \times k \times k$-cube which represents a square-based parallelepiped of the unit height. In order to build a cube, one needs to augment this parallelepiped with additional $k - 1$ parallelepipeds of the same size. As was shown above, $k^3 = k^2 \cdot 1 + k^2 \cdot (k-1) = k^2 \cdot 1 + \underbrace{k^2 \cdot 1 + k^2 \cdot 1 + \ldots + k^2 \cdot 1}_{(k-1)\, times}$; that is, we have a cube consisting of k identical parallelepipeds of the unit height.

Figure 2. A numeric environment of the multiplication table.

In that way, a computer spreadsheet that generates the multiplication table may become a thinking device for a student, thereby, bringing him or her to the deep structure of learning. However, the student's entry into this structure may turn to be unstable and the degree of its instability depends on the teacher's willingness or readiness, in turn, to enter the deep structure of teaching; in other words, it depends on a style of assistance a teacher is prepared to offer. If the student enters the deep structure of learning, but does not receive Style II assistance from the teacher (e.g., connecting the summation of the first n cubed integers to the sum of numbers in the $n \times n$ multiplication table or explaining a geometric meaning of the summation of numbers within a gnomon as a process of building a cube), it is quite likely that he or she would exit it back to the surface structure of learning.

Furthermore, receiving no support for intellectual curiosity affects one's cognitive disposition towards the continuation of having 'a good time' at the surface structure of

learning. This kind of a student's functioning within his or her universe is consistent with the dynamism of cognition expressed through the theoretical construct of the zone of proximal development [50]. The longer both the teacher and the student function at the deep structures of their universes; in other words, the longer Style II assistance for Type II application of technology is provided, the more concept learning can result from computational experiments.

8. Technology-enabled mathematics pedagogy

The notions of two types of technology application and two styles of assistance lead to the development of the concept of technology-enabled mathematics pedagogy (TEMP). This new concept can become a major pillar of modern signature pedagogy of mathematics as it can focus on the unity of mathematical experiment and mathematical proof. One of the major differences between TEMP and a mathematics pedagogy that does not incorporate technology pertains to the interplay between mathematical content under study and the scope of student population to which this content can be made available. Whereas problems that can be approached through TEMP may be fairly complex, using technology as a support system makes it possible to develop mathematical insight, facilitate conjecturing, and illuminate plausible problem-solving approaches to those problems. In comparison with mathematics pedagogy of pre-digital era that, in particular, lacks empirical support for conjectures, using TEMP has great potential to engage a much broader student population in significant mathematical explorations. TEMP provides teachers with tools and ideas conducive to engaging students in the project-based, exploratory learning of mathematics by dividing a project in several stages – empirical, speculative, formal, and reflective. Even if TEMP helps a student to reach the level of conjecturing without being able to proceed to the next level, its use is still justified.

By examining TEMP through the combined lens of teaching and learning, one can recognize significant merits of the pedagogy and its potential for achieving substantial learning outcomes. A student's entrance into deep structure of learning may be motivated by a sudden recognition of a mathematical concept that a computer supports, be it by a teacher's design or not. In the student's universe, the implicit structure of learning includes previous learning experiences and beliefs about what it means to learn and do mathematics [18]. Just as in the case of implicit structure of teaching, the implicit structure of learning affects both surface and deep structures of learning. An example of this relationship is a student's belief that any mathematical model, be it symbolic or iconic, serves only a single problem rather than multiple problems. Even if the same model emerges in different contexts, this belief prevents one from recognizing the sameness, affects one's desire to move from surface to deep structure of learning and, thereby, hinders conceptual understanding of mathematics. However, if a teacher functions at the deep structure of teaching, he or she can guide a student to understanding that just as different problem-solving strategies can be applied to a single problem, different problems may be resolved through a single approach. TEMP grounded in mathematical experiments does provide such an approach.

9. Collateral learning in the technological paradigm

9.1. Mathematical experiment and discovery by serendipity
John Dewey, the most influential contributor to the reform of American educational system in the first part of the twentieth century, made a case that a pedagogy which promotes and inspires learners' reflection on the material studied creates conditions for what he called

collateral learning – an activity which does not result from the immediate objective of the curriculum under study. Rather, the activity stems from a hidden domain of the curriculum. The following argument emphasizes the educational significance of collateral learning: "Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time" [17, p. 49]. TEMP requires from teachers deep knowledge of mathematics and proper understanding of how to integrate mathematics and technology in order to navigate students through obscure instances of collateral learning. Experimental mathematics approach is also conducive to "unintentional discovery" [26, p. 33], be it a new knowledge for a student, teacher, or professional mathematician. For a curious mind that continuously uses known concepts as the building blocks of unknown concepts to be discovered by serendipity, the use of a computer at a tool for experimentation with mathematics is conducive to the discovery of facts that were not expected to come about at the outset of a computational experiment. That is, computer-supported mathematical experiment is an essential mechanism for discovery learning.

9.2. Hidden mathematics curriculum

In a more general context, the notions of collateral learning and unintentional discovery bring to mind another educational construct known as hidden curriculum—"those nonacademic but educationally significant consequences of schooling that occur systematically" [30, p. 124]. This kind of learning experience is taking place within a context that is much broader than a topic of any given lesson and, through reflection, enables students to become aware of rules and guidelines typically associated with social relations and control of individual actions. The notion of hidden curriculum can be extended to include collateral learning and unintentional discovery that may take place within a pure academic domain when one is expected and even encouraged to make connections among seemingly disconnected ideas and concepts related to a specific subject matter. Thus, one can talk about hidden mathematics curriculum [2] – a didactic approach to the teaching of mathematics that motivates learning in a larger context that one "is studying at the time." Computing technology provides a learning environment to support this approach through which hidden messages of mathematics can be revealed to students by teachers as 'more knowledgeable others'. By the very design of a mathematical experiment and the nature of signature pedagogy of mathematical experiment, students are provided with ample opportunities for collateral learning and unintentional discovery as they develop mathematical habits of mind through continuous reflection on the results of a computational experiment. Several examples of using hidden mathematics curriculum framework to enable collateral learning are presented in the next few sections.

10. Comparing two types of experiments

A computational experiment made possible by TEMP can motivate the development of an analytic solution of a mathematical problem. The solution, in turn, can be modeled within another mathematical experiment and then the results of the two experiments – technology-enabled experiment and solution-enabled experiment – can be compared. Such comparison requires one's ability to correctly interpret different representations and use their similarity or identity to compare experimental results. A concept map between the two types of experiments is shown in Figure 3.

As an illustration, consider the derivation of quadratic formula for the equation $x^2 + x + c = 0$

$$(2)$$

where c is a real coefficient. Using the *Graphing Calculator 4.0* one can graph the locus of equation (2) shown in Figure 4. Any line $c = const$ has two points in common with the locus. In turn, any pair of such points with the coordinates x_1 and x_2 can be connected by a segment the mid point of which has the coordinates $(-\frac{1}{2}, c)$ whence $x_{1,2} = -\frac{1}{2} \pm \Delta$ where the value of Δ has to be determined. Furthermore, one can discover computationally that $x_1 \cdot x_2 = c$ and, therefore,

$$c = x_1 \cdot x_2 = (-\frac{1}{2} + \Delta)(-\frac{1}{2} - \Delta) = \frac{1}{4} - \Delta^2 \,,$$

whence $\Delta = \sqrt{\frac{1}{4} - c}$ and

$$x_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - c} \,. \tag{3}$$

In that way, formula (3) is a computationally developed quadratic formula that solves equation (2). It can be verified by graphing the relations

$$x = -\frac{1}{2} + \sqrt{\frac{1}{4} - c} \text{ and } x = -\frac{1}{2} - \sqrt{\frac{1}{4} - c}$$

in the plane (x, c). As a result, each of the last two relations would coincide graphically with the right and left branches of the locus of equation (2), respectively. That is, a quadratic formula may be obtained experimentally and then graphed as a way of verifying the correctness of the original experiment.
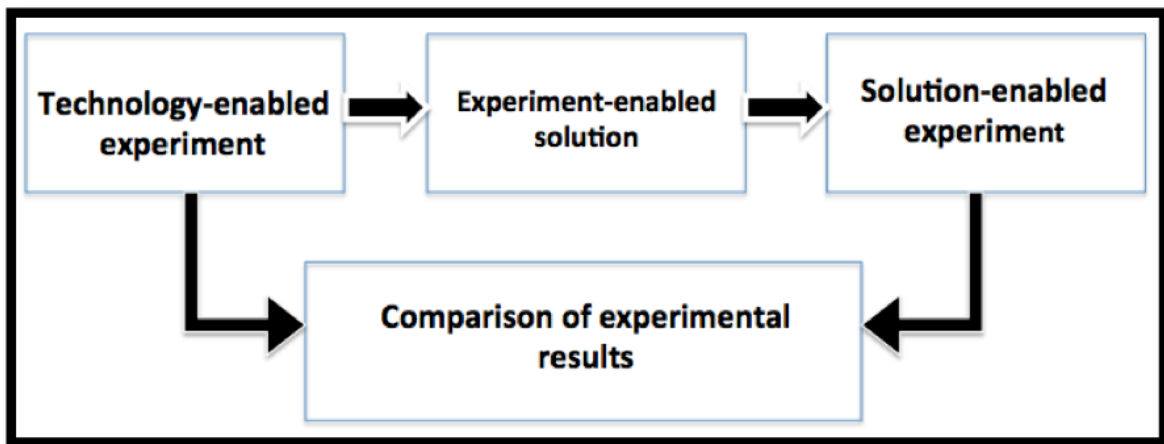

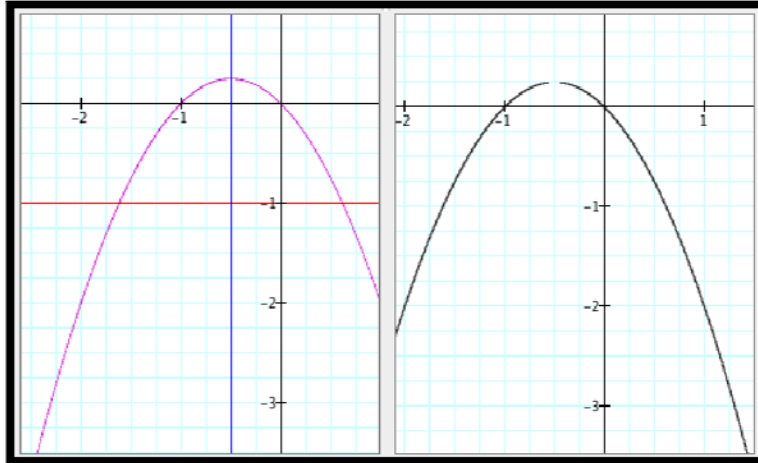Figure 3. Interplay between two types of experiment.

Figure 4. Loci of: left – equation (2) and right – formula (3).

11. Experiment informed by theory: An example

11.1. A historical investigation in the modern context

Consider the following question: *How can one arrange small square-shaped desks each of which seats four people (one at each side) in the form of rectangle so that the number of people seated around this rectangular desk is equal to the total number of small desks?*

This exploration can be carried out experimentally by using *The Geometer's Sketchpad* capable of constructing such rectangles (through trial and error) and calculating interactively the values of area (the number of desks) and perimeter (the number of people) in each case. With a relative ease, one can come up with two rectangles already known to Pythagoreans [49, p. 96]: 4×4 and 3×6 rectangles. However, experiment without theory is incomplete: it remains to be shown that there are indeed only two rectangles with such a property. To this end, a different environment created in the context of *The Geometer's Sketchpad* can be used to establish isomorphism between the construction of rectangles and covering fraction circle $\dfrac{1}{2}$ using two other fraction circles with no gaps or overlaps.

Indeed, if *x* and *y* are the (whole number) side lengths of a rectangle with the sought property, then $xy = 2x + 2y$ whence $\dfrac{xy}{2xy} = \dfrac{2x}{2xy} + \dfrac{2y}{2xy}$ or $\dfrac{1}{2} = \dfrac{1}{x} + \dfrac{1}{y}$. The last equation has the following geometric interpretation through fraction circles: one-half can be split into two equal parts, that is, into two one-fourths, and if the parts are not equal, one of them is bigger than one-fourth that could only be the fraction one-third. The latter, in combination with one-sixth creates one-half. That is, $\dfrac{1}{2} = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{3} + \dfrac{1}{6}$.

11.2. Theory evolves to inform experiment

Now one can explore a more difficult problem of finding rectangles with the four to one ratio of area to semi-perimeter. This time, finding such rectangles through pure experimentation in the context of *The Geometer's Sketchpad* (or any other dynamic geometry application) is quite a challenge. However, by using electronic fraction circles one can

discover that there exist three ways to split fraction circle $\dfrac{1}{4}$ into two fraction circles (Figure

5) and then show that the equation $\dfrac{1}{4} = \dfrac{1}{x} + \dfrac{1}{y}$ is equivalent to $xy = 4(x+y)$ by dividing

both sides of the latter equation by $4xy$. In that way, a theory developed around the partitioning of fraction circles in two other fraction circles informs experimental work with rectangles enabling one to know exactly the dimensions of the rectangles to be constructed. And when computer confirms that, indeed, the 8×8, 6×12 and 5×20 rectangles (Figure 6) have the four to one ratio of area to semi-perimeter, one not only has an informed experiment but the confirmation of theory through experiment as well.

The exploration with rectangles followed by the one with fraction circles that, in turn, facilitates a more complicated exploration with rectangles, illustrates an interplay that exists between experiment and theory. A simple experiment with the square-shaped desks by lacking full interpretation motivates the development of theory, which, in turn, informs other more technologically complex experiments. That is, the quest for internal validity of experiment becomes a basis for the development of theory and its subsequent external validation through application. Finally, the geometric results can be presented alternatively in

a graphic form by constructing the locus of a two-variable Diophantine equation $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n}$

with parameter n using the *Graphing Calculator* and demonstrating that when $n = 4$ the points (8, 8), (12, 6) and (20, 5) belong to the locus of the equation (Figure 7). Alternatively, this equation can be modeled numerically within a spreadsheet using inequalities as the means of achieving computational efficiency of the software [1].
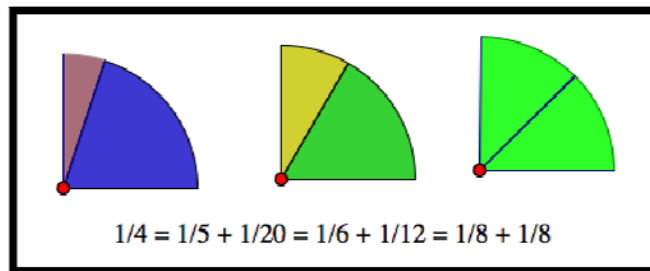

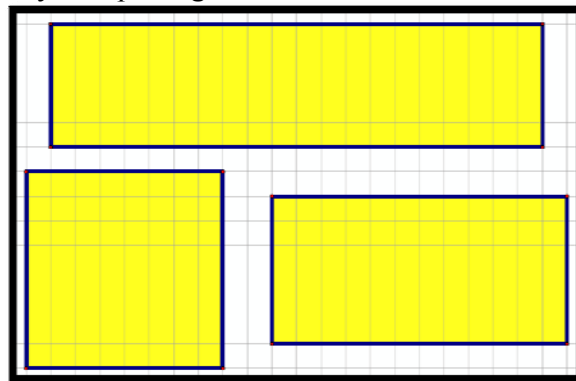Figure 5. Three ways of splitting fraction circle one-fourth in two fraction circles.

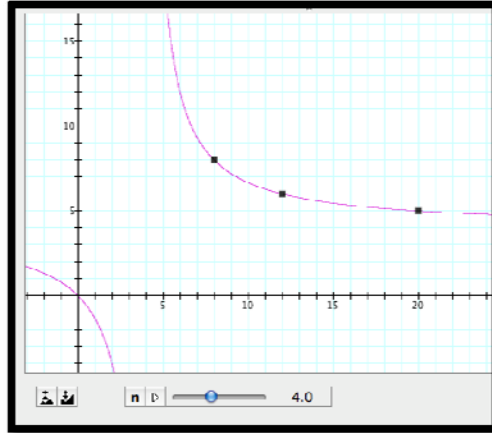Figure 6. Experiment informed by theory.

Figure 7. Graphical solution of the equation $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{4}$.

12. On the duality of experiment and theory

Experimental mathematics approach makes it possible to use uncomplicated context of pre-college mathematics curricula in order to illustrate a conceptually rich interplay between theory and computing. A conventional wisdom is that the use of technology facilitates access to complex ideas by hiding the complexity of mathematics within a computational environment. This section provides an example illustrating how theory (including paper-and-pencil computations) and computational experiment can complement each other.

Consider the case of solving in whole numbers the equation $\dfrac{1}{2} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$ which, in particular, describes all right rectangular prisms with integral dimensions x, y, and z, and volume being numerically equal to surface area. Indeed, for such a prism we have the equation $xyz = 2(xy + xz + yz)$. Dividing both sides of the latter equation by $2xyz$ yields the former equation. To solve this equation, note first that as shown in section 11.1, $\dfrac{1}{2} = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{3} + \dfrac{1}{6}$. In order to partition $\dfrac{1}{2}$ into three unit fractions, one has to partition each of the three fractions— $\dfrac{1}{3}$, $\dfrac{1}{4}$, $\dfrac{1}{6}$ — into two unit fractions. As

$$\frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{4} + \frac{1}{12}, \frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{1}{5} + \frac{1}{20} = \frac{1}{6} + \frac{1}{12},$$

$$\frac{1}{6} = \frac{1}{12} + \frac{1}{12} = \frac{1}{7} + \frac{1}{42} = \frac{1}{8} + \frac{1}{24} = \frac{1}{9} + \frac{1}{18} = \frac{1}{10} + \frac{1}{15},$$

the following nine solutions of the original equation result

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{5} + \frac{1}{20}, \quad \frac{1}{2} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12},$$

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{8}, \quad \frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{42},$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{8} + \frac{1}{24}, \quad \frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{18},$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{10} + \frac{1}{14}, \quad \frac{1}{2} = \frac{1}{3} + \frac{1}{12} + \frac{1}{12}, \quad \frac{1}{2} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}.$$

At the same time, one can observe (perhaps, after using a spreadsheet as shown in Figure 8) that the solution $\frac{1}{2} = \frac{1}{3} + \frac{1}{5} + \frac{1}{10}$ was not listed above. One may wonder: what is special about this missing representation? To answer this question, consider the equality $\frac{1}{2} = \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$ (see cells *C1*, *A2*, *C2* in Figure 8), which can be further transformed to the following representation of one-half as the sum of four unit fractions $\frac{1}{2} = \frac{1}{5} + \frac{1}{20} + \frac{1}{5} + \frac{1}{20}$. Due to the equality $\frac{1}{20} + \frac{1}{20} = \frac{1}{10}$, in which two unit fractions convolute into one, the missing representation, $\frac{1}{2} = \frac{1}{5} + \frac{1}{5} + \frac{1}{10}$, found through spreadsheet modeling has been discovered.

◇	A	B	C	D	E	F
1	y\x	3	4	5	6	
2	5		20	10		
3	6		12		6	
4	7	42				
5	8	24	8			
6	9	18				
7	10	15				
8	11					
9	12	12				

Figure 8. Ten representations of 1/2 as a sum of three unit fractions.

The case of a missing representation merits special consideration. It appears that the non-computational approach described above, although was based on a system, notwithstanding, had a flaw and, thereby, cannot be trusted. Without using technology, in order to overcome a possible deficiency of paper and pencil calculations, one has to continue partitioning fractions into the sums of four unit fractions to see if other cases when a sum of two unit fractions convolutes into one such a fraction could be found. This case demonstrates the didactical significance of the unity of computational and theoretical approaches in exploring mathematical ideas. Whereas one needs a theory in order to make sense of a

computational experiment, one also can benefit from the use of instructional computing as a means for the validation of theoretically developed results.

13. Validating mathematical experiment: An illustration

13.1. Internal and external validation of an experiment

   In education, any experiment can be validated through both internal and external means [10]. Internal means of validation of experiment comprise the basic set of skills and abilities needed for the interpretation of experimental results. In turn, informed interpretation creates conditions for generalizing from the experiment. It is through generalization that the experiment is validated externally. In the case of a computationally supported mathematical experiment, one has to possess some basic (grade appropriate) mathematical knowledge and skills in order to be able to interpret information generated through the experiment and, therefore, develop a more general perspective on this interpretation.

13.2. Fibonacci numbers as sums of binomial coefficients

   As an illustration, consider the following spreadsheet-based computational experiment that starts with generating partial sums of the series 1, 1, 1, 1, ... , that is, with generating the natural number series 1, 2, 3, 4, ... The two series are arranged in a two-column array as shown in Figure 9, columns A and B. The next step is to generate partial sums of natural numbers (triangular numbers, column C), than partial sums of triangular numbers (tetrahedral numbers, column D), than partial sums of tetrahedral numbers (pentatope numbers, column E), and so on. But each time, the first term of each new sequence is shifted down by two positions.

   The experiment continues by computing the sums of numbers in each row of the spreadsheet. These numbers are 2, 3, 5, 8, 13, 21, ... (column K). In order to grasp the intended meaning and appreciate the mathematical significance (in educational sense) of this computational experiment, one has to be able to recognize the nature of the numbers in column K comprising this final sum. Otherwise, that is, if these numbers are not recognized as Fibonacci numbers and the elements of the sums as the binomial coefficients C_{n-k}^{k}, one is unable to proceed from experiment to theory. This transition is needed to establish the external validity of the experiment that deals with the issue of generalizability. In this case, the theoretical justification of the experiment consists in formal demonstration of the emerging conjecture, namely, that

$$C_n^0 + C_{n-1}^1 + C_{n-2}^2 + ... + C_{n-r}^r = F_n, \, n = 2, 3, 4, ... , \text{ where } r = \lceil n/2 \rceil. \quad (4)$$

Here $\lceil x \rceil$ denotes the greatest integer smaller than or equal to *x*.

	A	B	C	D	E	F	G	H	I	J	K
1	1	1									2
2	1	2									3
3	1	3	1								5
4	1	4	3								8
5	1	5	6	1							13
6	1	6	10	4							21
7	1	7	15	10	1						34
8	1	8	21	20	5						55
9	1	9	28	35	15	1					89
10	1	10	36	56	35	6					144
11	1	11	45	84	70	21	1				233

Figure 9. Generating successive partial sums.

13.3. Developing situational referents for Fibonacci numbers

Therefore, if one does not possess mathematical knowledge necessary for establishing the internal validity of an experiment, he or she is not prepared for the discussion of its external validity, or, at the very least, any effort in establishing the external validity of an experiment may go astray. By connecting computational experiment to a real-life context such knowledge can be developed. Considering "every obstacle an opportunity for the exercise of ingenuity instead of an insuperable barrier" (McCall, 1923, p. 7), one can precede a somewhat abstract computational experiment with a more concrete hands-on activity involving familiar objects. According to Freudenthal [20], "It is independency of new experiments that enhances credibility ... [for] repeating does not create new evidence, which in fact is successfully aspired to by independent experiments" (pp. 193-194). Having different representations of an experimentally developed concept makes it easier to establish internal validity of experiment. In doing so, one can simplify the abstractness of symbols in relation (4) by introducing a context from which relation (4) can be independently developed. The following problem can provide such a context to serve as a situational referent for Fibonacci numbers.

Buildings of different number of stories are given and one has to paint them with a single color in such a way that no two consecutive stories are painted with it. How many ways of such painting of one, two, three, four, etc.–storied buildings are possible? Note: not painting a building at all is considered a special case of painting as in that case the main condition of not having consecutive stories painted is satisfied.

Without using technology, it is not difficult to conclude that the number of different paintings of one, two, three and four-storied buildings is equal, respectively, to 2, 3, 5, and 8. In general, there exist C_{n-k}^{k} ways to paint an $(n-1)$-story building so that exactly k non-adjacent stories are painted with one color. Indeed, C_{n-k}^{k} represents the number of ways to choose k objects (e.g., stories) out of $n-k$ objects (stories), where $0 \leq k \leq n$. In order to separate k painted stories, one needs to insert $(k-1)$ additional stories. In that way, a $(n-k)$-storied building turns into the building with $(n-k)+(k-1)=n-1$ stories in which k non-

adjacent stories are painted. Therefore, a ($n - 1$)-story building can be painted in $\sum\limits_{k=0}^{\lceil n/2 \rceil} C_{n-k}^k$

ways. In particular, a one-story building (n = 2) can be painted in

$\sum\limits_{k=0}^{\lceil 2/2 \rceil} C_{2-k}^k = C_2^0 + C_1^1 = 1 + 1 = 2$ ways, a two-story building (n = 3) can be painted in

$\sum\limits_{k=0}^{\lceil 3/2 \rceil} C_{3-k}^k = C_3^0 + C_2^1 + C_1^2 = 1 + 2 + 0 = 3$ ways, a three-story building (n = 4) can be painted in

$\sum\limits_{k=0}^{\lceil 4/2 \rceil} C_{4-k}^k = C_4^0 + C_3^1 + C_2^2 = 1 + 3 + 1 = 5$ ways, and a four-story building (n = 5) can be painted

in $\sum\limits_{k=0}^{\lceil 5/2 \rceil} C_{4-k}^k = C_5^0 + C_4^1 + C_3^2 + C_2^3 = 1 + 4 + 3 + 0 = 8$ ways. Note that 2, 3, 5, and 8 are

consecutive Fibonacci numbers.

13.4. Mathematical proof as external validation of the experiment

Now, one can proceed to proving identity (4) which connects combinations to

Fibonacci numbers. Following [4], consider the function $F(x) = \sum\limits_{n=0}^{\infty} F_n x^n$, where the numbers

F_n are defined by recurrence relation (1). Noting that

$$\sum_{n=2}^{\infty} F_{n-1} x^{n-1} = \sum_{n=1}^{\infty} F_n x^n = F(x) - F_0 x^0 = F(x) - 1 \text{ and } \sum_{n=2}^{\infty} F_{n-2} x^{n-2} = \sum_{n=0}^{\infty} F_n x^n = F(x),$$

one can write

$$F(x) = F_0 x^0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n = 1 + x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n = 1 + x + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}$$

$$= 1 + x + x[F(x) - 1] + x^2 F(x) = 1 + xF(x) + x^2 F(x),$$

from where the generating function for Fibonacci numbers results

$$F(x) = \frac{1}{1 - (x + x^2)} \ .$$

Now, using the rule of summation of geometric series, the binomial expansion

$$(1 + x)^k = \sum_{r=0}^{k} C_k^r x^r \ ,$$

and the substitution $n = k + r$ yields

$$F(x) = \frac{1}{1 - (x + x^2)} = \sum_{k=0}^{\infty} (x + x^2)^k = \sum_{k=0}^{\infty} x^k (1 + x)^k = \sum_{k=0}^{\infty} x^k \sum_{r=0}^{k} C_k^r x^r = \sum_{n=0}^{\infty} x^n \sum_{r=0}^{\lceil n/2 \rceil} C_{n-r}^r \ .$$

Finally, from the uniqueness of the power series expansion about x = 0 the equality

$F_n = \sum\limits_{r=0}^{\lceil n/2 \rceil} C_{n-r}^r$ results, which is an alternative way of writing relation (4).

13.5. From mathematical experiment to an unsolved problem

Consider Figure 9. If one uses the strings of numbers displayed in rows 1, 2, 3, 4, ..., 11 of the spreadsheet as coefficients in the powers of *x*, one can construct the polynomials

$$P_1(x) = x+1, \ P_2(x) = x+2, \ P_3(x) = x^2+3x+1, \ P_4(x) = x^2+4x+3, ...,$$

$$P_{11}(x) = x^6+11x^5+45x^4+84x^3+70x^2+21x+1,$$

called Fibonacci-like polynomials [5] and defined in the general form as

$$P_n(x) = x^{\mathrm{mod}(n,2)}P_{n-1}(x)+P_{n-2}(x), \ n \ge 2, \ P_0(x) = 1, \ P_1(x) = x+1,$$

where $\mathrm{mod}(n,2)$ is the remainder of *n* divided by 2. An interesting, in fact, remarkable, property of these polynomials is that they don't have complex roots for any $n \in N$. This property was established [5] computationally using *Maple* for $n \le 100$ and it remains an open problem in mathematics. This example shows how the mathematical experiment approach applied to the context of a well-known and thoroughly studied entity of mathematics, Fibonacci numbers, can open a window to a mathematical frontier and thus bridge mathematics education and mathematics research.

15. Concluding remarks

At the focus of this paper was the notion of a mathematical experiment in the technological paradigm as signature pedagogy of mathematics at the pre-college level. A number of software products such a an electronic spreadsheet, the *Graphing Calculator*, *Maple*, *Wolfram Alpha*, and *The Geometer's Sketchpad* was suggested as appropriate tools for mathematical experimentation. The paper reviewed mathematics education literature and the modern day educational standards related to the use of computers for teaching mathematics in the schools. It connected pioneering ideas by Euler about mathematical experiments with numbers to the use of the word experiment in grade-appropriate mathematics teaching practices. The paper highlighted the notions of collateral learning, unintentional discovery, and hidden mathematics curriculum as essential mechanisms of technology-enabled mathematics pedagogy. The relationship between the so-called technology-enabled and solution-enabled experiments was discussed in the context of experimentation with parameter-dependent quadratic equations. The classic context of pre-college mathematics was used to illustrate how a mathematical experiment could be first validated internally through the interpretation of numeric results in order to provide a basis for moving to its external validation at the symbolic level through doing a formal mathematical proof. The final idea of the paper was to demonstrate how a mathematical experiment at the pre-college level could be extended to open a window to a mathematical frontier and formulate an unsolved problem in mathematics about the roots of polynomials associated with Fibonacci numbers.

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