

TECHNOLOGY-IMMUNE/TECHNOLOGY-ENABLED MATHEMATICAL PROBLEM SOLVING AS INSTRUMENTAL GENESIS

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Abstract. This paper demonstrates diverse uses of mutually-complementary digital tools through the perspective of technology-immune/technology-enabled K-16 problem solving and compares this perspective with the theory of instrumental genesis. The demonstration and comparison take place over the background of the modern-day mathematics teacher education. The ideas of the theory of instrumental genesis are extended to include multiple instruments, multiple subjects, and multiple objects in the context of a single, yet open-ended mathematical task. An interplay between basic and advanced instrumentations (alternatively, technology-immune activities) is considered in the context of using multiple instruments. Mathematical models that support the discussion include a system of two linear algebraic equations in two unknowns, a non-linear indeterminate algebraic equation in two unknowns, a two parametric linear difference equation of the second order and its non-linear recursive modification.

Key words: mathematics, technology, problem solving, problem posing, instrumental genesis, TITE methodology, teacher education, educational psychology

1. Introduction

The aim of this paper is to connect the notion of technology-immune/technology-enabled (TITE) problem solving originally introduced in [3] as an extension of Type II uses of technology framework [33] to the theory of instrumental genesis [37] with its origin in the seminal ideas by Vygotsky [44] about mediating cognitive processes, learning included, by technical devices and psychological tools. Solving a TITE mathematical problem cannot be outsourced entirely to a technical device capable of diverse numeric and/or symbolic computations, yet the role of this device in dealing with the problem is critical. A TITE mathematical problem solving may include the use of multiple software products (digital devices) in support of a single task. In many cases, using different tools – numeric, symbolic, graphic – not only provides support in solving a problem but it motivates one’s deep thinking

about computationally obtained results that can lead to posing a new problem or a family of problems.

Due to the advent of computers in education, the theory of instrumental genesis is used by researchers to study how new pedagogical ideas about the appropriation of an artifact as a material object and its elaboration to become an instrument as a psychological concept develop. After an artifact is turned into an instrument, the latter becomes inserted between the subject (user of digital technology) and the object (task) and can act bi-directionally; that is, affecting both the completion of the task and intellectual development of the user. According to Vygotsky [44], “the instrumental method distinguishes a twofold relation between behavior and an external phenomenon [which] ... can play the role of the object toward which the act of behavior is directed ... [and] of a means by means of which we direct and realize the psychological operations (memorizing, comparing, selecting, etc.) necessary for the solution of the problem.” By acting towards the task, an instrument evinces pragmatic mediation and by acting towards the user it supports epistemic mediation [31]. This duality of the role of an instrument in problem solving allows one to see the TITE concept through the lenses of the theory of instrumental genesis. Indeed, whereas during a technology-enabled stage of activity an instrument extends mental capacity of a subject, during a technology-immune stage of activity, by transforming an artifact into an instrument, the subject grows in his/her cognitive power.

This paper presents a variety of examples from different grade levels of mathematics to elucidate a close relationship between two educational perspectives on learning in the digital era. While some examples will be mentioned briefly in making different points, three illustrations of using multiple instruments in the context of a single problem will be described in detail. Mathematical models associated with these illustrations include a system of two linear equations with two unknowns (with the origin at the elementary level), a non-linear indeterminate equation in two variables (stemming from the middle or secondary school levels), and a linear two parametric difference equation of the second order (arising at the secondary or tertiary levels). Using software tools as diverse as *Kid Pix* [10], *The Geometer's Sketchpad* [26] (referred to below as *GSP*), the *Graphing Calculator* [8] (referred to below as *GC*), *Wolfram Alpha* (<https://www.wolframalpha.com>; referred to below as *WA*), *Maple* [15], and Excel spreadsheet, both similarities and differences between TITE problem solving methodology and instrumental genesis theory in the context of mathematics teacher education will be demonstrated. Prospective teachers' reflections on the use of digital instruments, both at the basic and advanced levels, as means of learning mathematics for teaching are included in the section on teacher education. Finally, the paper will emphasize the important role of mutually-complementary digital instruments in promoting experiment-oriented methods of teaching mathematics in the modern-day educational environment.

2. Two types of technology application in education

Despite all technological innovations available for teaching and learning in the 21st century, the pedagogy of a mathematics classroom might still be found focusing on drill and practice (D&P). In arithmetic, D&P may include placing right answers in boxes in response to addition, subtraction, multiplication, or division queries to be either accepted or rejected by a computer. In geometry, D&P may include the construction of basic shapes or comparing their areas through computational measurement (e.g., stamping the correct shape in response to a quest for selecting a specific shape from several shapes available or considering numeric evidence as proof of congruency when a triangle is cut in two parts by a median). In computer-assisted algebra, D&P may include using trial and error in finding roots of

quadratic equations to be approved by a computer, factoring polynomials, or inverting functions and matrices with the help of software.

In the early 1980s, at the outset of using computers in the schools, Maddux [33] referred to teaching through D&P in the general context of education as Type I uses of technology and advocated for using technology in support of “new and *better* ways of teaching” (p. 38, italics in the original). He called such new ways of integrating computer-assisted didactics as Type II applications of technology. The Type I vs. Type II distinction proved to be pedagogically effective as two decades later, a book edited by Maddux and Johnson [34] included diverse classroom experiences specifically reflecting on successful Type II computer-assisted pedagogies. In mathematics education, solving and posing multistep problems, using mathematical concepts as tools in computing applications, or exploring curricular topics that otherwise are not grade-appropriate (or simply unattainable) are examples of Type II computer-assisted pedagogies.

The notion of the two types of technology applications is not only a pedagogically useful distinction between different perspectives on technological mediation. This notion highlights the dichotomy between instructivist and constructivist learning environments [11] that are, as far as mathematics is concerned, aimed at, respectively, nurturing procedural skills and developing conceptual understanding of the skills as the foundation of procedural competence. Whereas instructivist learning environments are very specific and quite limited in terms of their learning outcomes, constructivist learning environments help learners to develop the so-called epistemic fluency [35] – the ability to use productive thinking [51] and conceptual shortcuts [14] as means of conceptual knowledge construction and cultivation of computational competence. Epistemic fluency can be both epistemically mediated by an instrument developed from an artifact and expertly used in turning an artifact into an instrument. Just as the boring and clumsy uses of mathematics could (and should) motivate the development of its more procedurally effective concepts (e.g., addition vs. counting, multiplication vs. repeated addition, decimals vs. common fractions), one’s mundane experience with Type I application of technology can motivate the epistemic development of creative ideas to support educational applications of Type II.

One can observe that the growth in symbolic computation capabilities of mathematical modeling software makes the outcome of problem solving dependent on “an *automatic transport phenomenon*” [23, p. 205, italics in the original] – the subject’s ability to correctly input all the problem’s data into a computer. This phenomenon also blurs the dichotomy between Type I and Type II technology applications. In order to continue securing benefits from Maddux’s [33] educational framework, the notion of Type II uses of technology has to be elevated to a higher level where one deals with TITE problems. One can say that the TITE problem-solving methodology is the Type II application of technology of the second order.

3. TITE methodology as instrumental genesis

As was mentioned in the introduction, the theory of instrumental genesis has been used by educational researchers to study how new pedagogical ideas about the appropriation of an artifact as a *material object* and its elaboration to become an instrument as a *psychological concept* in support of carrying out a particular task take shape. In the modern era, computers are commonly used as technical devices and the symbolic language of mathematics serves as a psychological tool that mediates thinking as a high-level mental activity associated with the use of computers. In terms of the theory of instrumental genesis, technology-immune (TI) part of problem solving may be construed as “*instrumentation*” – a process through which a subject (i.e., a user of a digital tool) develops intellectually (e.g., by

selecting appropriate mathematical concepts to be used in computing applications); its technology-enabled (TE) part may be construed as “*instrumentalization*” – a process through which an artifact (i.e., a digital tool) broadens the realm of utilization. As Lonchamp put it, “In the instrumentation process, the subject develops, while in the instrumentalization process, the artifact evolves” [31, p. 216]. Within a TITE activity, its TI part represents the subject’s thinking about how to use software as an instrument which mediates problem solving – this can be seen as the process of instrumentation because the use of software (high-tech artifact) shapes this thinking. By the same token, a TE part represents the case when software, after being transformed into an instrument, enables the process of instrumentalization due to which the resolution of a particular problem/situation can be computationally enhanced as a new application of software has been developed. Just as TI mathematical activities can differ in complexity on the spectrum from basic to advanced, one may distinguish between basic and advanced instrumentations as a subject turns an artifact into an instrument.

For example, one can use a spreadsheet to generate consecutive square numbers 1, 4, 9, 16, 25, ... by entering these numbers one by one into the cells of the spreadsheet or as squares of the corresponding natural numbers entered, perhaps recursively, into an adjacent column or row. This action may be considered as a basic instrumentation in operating a spreadsheet. Alternatively, the square numbers can be generated as partial sums of consecutive odd numbers 1, 3, 5, 7, 9, It is the relationship between two sets of integers (odd and square numbers) in the form $n^2 = (n - 1)^2 + (2n - 1)$, or, alternatively, $n^2 = 1 + 3 + 5 + \dots + (2n - 1)$, that is inserted as a psychological tool between the subject and the spreadsheet. This TI-type action by the subject can be characterized as an advanced instrumentation for it requires knowledge of (although quite obvious) relation between two consecutive squares expressed through a diagram of Figure 1. In particular, the diagram shows how augmenting a square by a gnomon (which geometrically represents an odd number as a double plus one) results in the next (integer sided) square.

By the same token, the capability of a spreadsheet to accommodate this and other mathematical relations and formulas through the technique of cell referencing makes it possible to use this computational tool, originally developed for activities outside of (mathematics) education, as a medium of demonstration through numerical evidence how different sets of integers are related. That is, integrating mathematical knowledge (either basic or advanced) and a spreadsheet constitutes a TE part (or instrumentalization) of the TITE methodology when a spreadsheet becomes an artifact turned into an instrument in order to be used outside the domain of its originally intended agency.

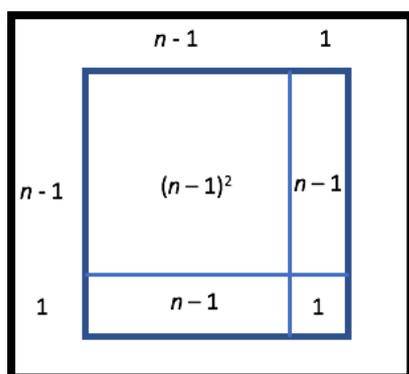


Figure 1. Diagrammatic representation of algebraic formula.

A TITE mathematical problem solving may include the use of multiple artifacts (digital devices) in support of a single task. In the case when the devices are complementary such use of technology may be considered through the lenses of double stimulation for a subject who is assigned a specific task or extends the assignment in a computational discovery fashion. The method of double stimulation is based on the idea of “using two sets of stimuli, one the primary set that has to be mastered and the other an auxiliary set that

can serve as an instrument for mastering the primary set” [32, p. 47]. For example, in deciding the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ – the first stimulus – a subject can first extend the spreadsheet filled with consecutive squares to numerically model partial sums of their reciprocals (instrumentalization) to see the apparent convergence of the partial sums to the number 1.6449... . Then, using WA, the subject can find out that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (identity known as the Basel Problem solved by Euler in 1735) and compare (instrumentation) a decimal approximation of $\frac{\pi^2}{6}$ to the decimals generated by the spreadsheet. That is, the second set of stimuli may be referred to as an integrated spreadsheet [4] as it consists of two rather distinct devices having complementary computational capability – numeric and symbolic. One can see how integrating commonly available digital instruments can be used in mastering the decision about the convergence of a series, thus indicating that, indeed, “double stimulation is not a mechanical process; the prevailing instruments and procedures have to be creatively adapted for solving the problems encountered” [39, p. 245]. Furthermore, the presence of the number ρ , which typically is associated with circular functions, may motivate one to look into the nature of the sums of the reciprocals of squares of integers and find out (the instrumentation) how great Euler solved the Basel Problem [19]. One can see that using two digital tools not only provides support in solving a problem (the first stimulus working in a TE fashion or enabling instrumentalization) but it can motivate one’s deep thinking (belonging to the second set of stimuli working epistemically in a TI fashion or enabling instrumentation) about computationally obtained results.

4. Multiple levels of instrumentation call for a new curriculum

A point can be made that as the sophistication of digital tools progresses, mathematical problem-solving practices with the use of technology, that until recently were considered among Type II pedagogies, become less and less intellectually engaging. This reduction in complexity of doing mathematics in the digital era blurs the distinction between the two types. Put another way, a digital tool with highly sophisticated symbolic computation capabilities enables mathematical problem solving to be reduced to a simple push of a button (e.g., using WA in factoring the polynomial $x^3 - 2x^2 - 2x - 3$, or solving the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$ in positive integers as shown in Figures 2 and 3, respectively; although the latter case does require being specific about the nature of x and y before asking the tool to solve the equation by clicking at the equal sign in its input box).

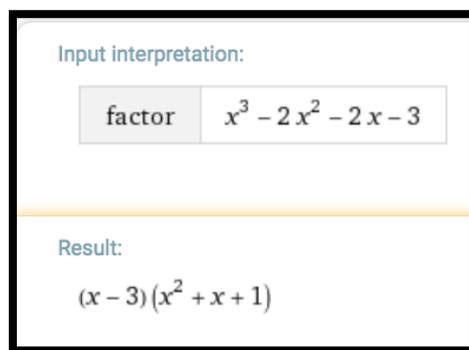


Figure 2. Factoring polynomial with WA.

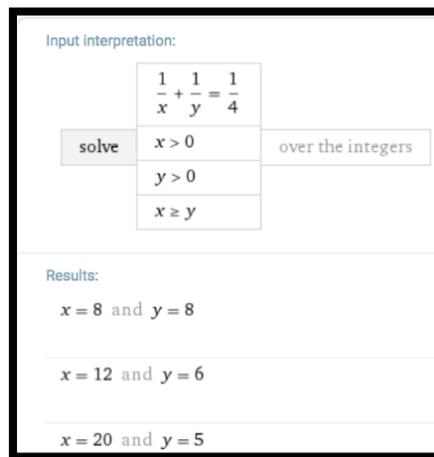


Figure 3. Solving an indeterminate equation with WA.

It should be noted that the type of technology usage or the level of instrumentation associated with a particular mathematical activity may depend on what kind of tool is selected to support the activity. For example, in the context of the *GSP* the construction of, say, the point (1, 1) requires just typing the coordinates in the Plot Points dialogue box of the Graph menu of the program. The activity is purely of a TE type and the program is used as an instrument requiring only the very basic level of instrumentation. Yet, in the context of the *GC*, whereas such construction is possible, it requires digital fabrication [12, 22, 49] of the point through graphing the system of inequalities $|x - 1| < \varepsilon$, $|y - 1| < \varepsilon$ for a sufficiently small ε , something that represents the point in the form of a tiny square of side 2ε . If one wants to digitally fabricate the point in the form of a tiny disk of radius ε , one has to graph the inequality $(x-1)^2 + (y-1)^2 < \varepsilon$; that is, to graph a set of points located inside a circle of radius ε centered at the point (1, 1).

One can see that the use of the *GSP* in plotting a point falls into Type I category, alternatively, requiring basic instrumentation, and digitally fabricating the point in the context of the *GC* is an example of a Type II use of technology of the second order for it requires not only knowledge of two-variable inequalities but the application of this knowledge through the process of digital fabrication. In other words, the use of the *GC* in plotting a point through a TITE problem-solving methodology can be referred to as turning an artifact into an instrument requiring an advanced instrumentation. Furthermore, one can characterize the TITE digital fabrication of geometric objects in the context of the *GC* as instrumentalization – a process through which this computer application broadens the realm of utilization by defying the perceived constraints, typical for artifacts, through revealing, for a novice user, hidden, yet available, psychological operations.

The above examples can demonstrate that whereas in the context of a single tool, Type I and Type II applications may become indistinguishable, in the context of more than one tool, Type I may become Type II and vice versa. Alternatively, depending on a tool, a TE activity (or instrumentalization), to be successful, requires augmentation by a TI component (or advanced instrumentation). Not only a complex TITE activity can be seen through the lenses of instrumental genesis, but such pedagogically diverse affordability of high-tech software products may call for a new mathematics curriculum which allows for the unity of cognitive engagement and technology use as essential elements of mathematical problem solving. This new curriculum should take into consideration both positive (conceptual development) and negative (button pushing) affordances of computer technology. Put another way, the need for mathematical problem solving supported by the instrumental perspective on

TITE methodology motivates the development of new pedagogical ideas for problem posing [5].

As was already mentioned in the introduction, the theory of instrumental genesis distinguishes between pragmatic mediation when an instrument is used by a subject to reach a certain goal and epistemic mediation when through the use of an instrument the subject develops intellectually [31]. When the goal is to solve a mathematical problem using an instrument, one can talk about pragmatic mediation by the instrument. Posing a similar problem or modifying the original one using this instrument requires insight into the problem situation and understanding of how the instrument was used. In the case of problem posing, the instrument is directed toward the subject for whom it provides epistemic mediation. In other words, epistemic mediation is “a deliberate process of deepening inquiry by creating external knowledge artifacts that ... provide stepping stones for directing and guiding further personal or collective inquiry efforts” [39, p. 244]. One can say that pragmatic mediation represents a TE part of an activity and epistemic mediation represents a TI part of the activity. For example, in the case of using an instrument by a subject in the context of a computational experiment, an instrument, by pragmatically mediating the experiment, delivers information and this can be seen as a TE part of the activity. But experimental data has to be explained and, through a TI part of the activity, an epistemic mediation takes place as a way of seeking explanation of the results of the experiment. Put another way, at the beginning of TITE problem solving, pragmatic mediation leads to the delivery of numeric, graphic, or symbolic information. Then, epistemic mediation comes into play as a reflection on problem solving to enable new problems to be formulated. The pioneering ideas by Isaacs [25] about information vs. explanation cognitive paradigm can nowadays be described in terms of an interplay between pragmatic and epistemic mediation.

5. An incorrect use of an instrument prompts a modification of instrumentation

Verillon and Rabardel provided what they called “a simplified example of instrumental genesis: a baby learning to use a spoon ... [a process through which] he acquires some knowledge about the behavior of liquids [e.g., milk] as opposed, say, to mashed potatoes” [43, p. 85]. These authors used this context as an example of learning through instrumentation when a spoon from an artifact becomes an instrument inserted between the source of food and the baby and it acts bi-directionally. This example can also demonstrate a negative affordance of an instrument, something that may not necessarily be appreciated by a subject. Indeed, a baby may simply not care whether milk stays in the spoon or is spilled over on its way to his mouth. While a baby may learn, as these authors suggested, that a spoon is not the right tool to carry milk, infants frequently do not care of the outcome. Nonetheless, baby develops some kind of understanding of the negative affordance of a spoon-based instrument as a carrier for milk, especially when being supervised or assisted by an adult (see a citation from Vygotsky [48] at the end of the paper).

Somewhat similarly, yet in a much more advanced learning situation, a student of undergraduate mathematics may use a spreadsheet to decide whether the harmonic series converges or diverges. The spreadsheet is programmed (basic instrumentation) to generate natural numbers, their reciprocals, and partial sums of the reciprocals within three columns, respectively. Because of an extremely slow growth of the partial sums, the so constructed spreadsheet (an artifact turned into an instrument) would not demonstrate the divergence of the harmonic series and, thereby, this computational experiment is likely to lead the student towards a wrong conclusion. However, spreadsheet is not spoon and series is not milk. That is why, an assistance of a ‘more knowledgeable other’ is needed for mathematical learning to take place.

With this in mind, as shown elsewhere [1], in order to demonstrate the divergence of the harmonic series, one can still use a spreadsheet, but its instrumentalization takes a very different path. It requires creating a two-dimensional modeling environment based on the knowledge of mathematics and spreadsheets. The mathematics part includes knowing Bolzano-Cauchy principle of convergence of a sequence which states that in order for the variable x_n (in our case, $x_n = \sum_{i=1}^n \frac{1}{i}$ – the sequence of partial sums of the harmonic series) to have a final limit it is necessary and sufficient that for any $\varepsilon > 0$ there exists $N > 0$ such that the inequality $|x_n - x_m| < \varepsilon$ holds true for all $n > N, m > N$. The spreadsheet part includes knowing the function INDEX, a special, not commonly known computing instrument. A similar example of the negative affordance of an instrument deals with the negative affordance of a hand-held graphing calculator as a means of exploring the limiting behavior of the function $y = \ln x + 10 \sin x$ as x grows larger through graphing [18], although it appears that, unlike a spreadsheet, the calculator cannot be modified to enable an alternative instrumentation with a positive affordance outcome. To a certain extent, the calculator's representation of the logarithmic function at the infinity is reminiscent of baby's use of a spoon in carrying milk or any other liquid, in general.

6. Managing information provided by the modern-day artifacts

A TI part of a TITE activity can be at least of two types. The first type is when one uses mathematical concepts as tools in computing applications with the goal to turn an artifact into an instrument. One such example is the above-mentioned use of Bolzano-Cauchy principle of convergence in the context of a spreadsheet. The second type of a TI part of a TITE activity, depending on an instrument used, may include the need to analyze and make sense of numeric, symbolic, graphic, or even verbal information. For example, a spreadsheet can generate Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, ... , and the use of the greatest common divisor function would demonstrate numerically that two consecutive Fibonacci numbers are relatively prime. This TE part of the activity can motivate a follow up explanation of this phenomenon in a TI fashion. In particular, one can connect Fibonacci recursion to the Euclidean algorithm. Indeed, by interpreting the sequence of relations $34 = 21 + 13$, $21 = 13 + 8$, ..., $2 = 1 + 1$, as representing the number in the left-hand side being the dividend, the larger and the smaller numbers in the right-hand side being, respectively, the divisor multiplied by the quotient 1, and the remainder, one can see that having the number 1 as the last non-zero remainder, proves (due to Euclid) the relative primality of 34 and 21. Without loss of generality, this observation can be extended to any two consecutive Fibonacci numbers. However, the use of other tools in this context may lead to some unexpected instrumental genesis outcomes.

For instance, one can also enter the first few Fibonacci numbers into the input box of WA which immediately offers the recurrence relation and the generating function for Fibonacci numbers. While the term generating function sounds enticing, the very concept may be beyond the grade level of the subject involved. In that way, WA remains an artifact for the subject that would not act as a grade-appropriate instrumental psychological concept. Likewise, the Online Encyclopedia of Integer Sequences (OEIS[®]) as an artifact offers, among many other things, the continuation of the first few terms in the form 1, 1, 2, 3, 5, 8, 13, 39, 124, ... interpreting each term as the sum of the preceding two terms with the digits reversed. This unexpected interpretation by OEIS[®] might turn an artifact into an instrument if one realizes that $39 = 31 + 8$ and $124 = 93 + 31$. An interesting instrumental genesis activity motivated by the above instrumentation, which is exactly of the TITE kind, is to program a spreadsheet to generate this sequence. Such variety of concepts that different artifacts make

available at the pre-instrumental level raises an issue of how a subject can manage the abundance of information with such an easy modern-day access [17]. That is, a TITE activity can provide either instrument-based or artifact-based information that may not be originally expected to be obtained and thus cannot be easily understood, let alone meaningfully interpreted, by a subject.

7. TITE methodology illustrations as instrumental genesis

7.1. Illustration 1: From early algebra to Cramer's Rule with multiple instruments

Often, solving a word problem from the secondary school mathematics curriculum involves constructing a system of two linear equations in two unknowns and using algebraic transformations to find the unknowns. Problems of that kind can be reformulated in terms of grade-appropriate context and then introduced already at the early elementary level. The instrumental perspective on learning mathematics in the digital era makes it possible to insert technology between a student and a problem. For example, a student at that grade level can be asked to solve a (mathematical) word problem by trial and error using computer program *Kid Pix* – software for creative activities of young children. One can say that, in general, *Kid Pix* is already an instrument ready to be used by children, for example, by creating images of different objects, either available as stamps in the tool kit of the program or simply drawn using the program's multi-colored pencil. However, in order to be used for mathematical problem solving a child first encounters the program as an artifact and gradually transforms it into an instrument under the guidance of a teacher.

7.1.1. Early algebra through the method of double stimulation

As an illustration when *Kid Pix* mediates problem solving through the method of double stimulation, consider the following problem [2]:

Five drimps and one grimp have the total of 16 legs. Four drimps and three grimps have the total of 15 legs. How many legs does a drimp have and how many legs does a grimp have?

Here, the use of the names drimp and grimp serve as a vicarious form for variables. In other words, the artificial creatures serve for a child as the first set of stimuli to deal with. *Kid Pix* serves as the second set of stimuli allowing for drawing pictures of the first set. A child, for example, can select (by drawing) drimp to have two legs and grimp to have four legs and then count the total number of legs among them in the double stimulation fashion. The child would end up with 14 legs and would have to make a different selection as this trial ends up in error. What if we have a three-legged drimp? Then, by drawing five three-legged drimps the child would count 15 legs leaving just one leg for the remaining one grimp. This trial works for the first condition. Could this combination, Drimp = 3 and Grimp = 1 work for the second condition? It can, as four three-legged drimps have 12 legs and three one-legged grimps have 3 legs thus having 15 legs among them. The problem has been solved (Figure 4). Without realizing it, the child managed to solve a system of two linear equations with two unknowns as a trial and error physical activity by partitioning 16 objects into five groups of one cardinality and a single group of another cardinality, and then, using the same cardinalities, by partitioning 15 objects into four and three such groups.

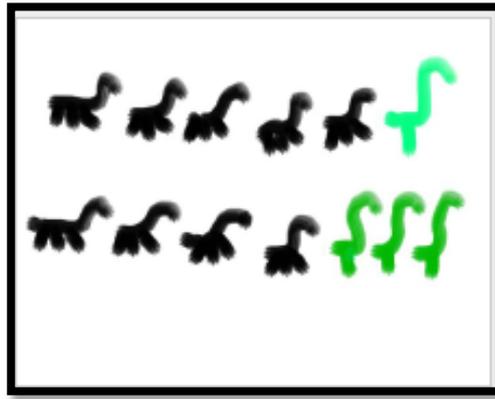


Figure 4. Using *Kid Pix* as an instrument.

7.1.2. Discovering conceptual bond

Now, a child is ready to move from the instrument-created solution to its description through numbers and words. When such a move is encouraged by a teacher, it reflects the position by Vygotsky regarding instruction which “*is only useful when it moves ahead of development ... [as it] impels or wakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development*” [47, p. 212, italics in the original]. Working within this zone, the child can generate the following description of the images using “the second order symbolism which involves the creation of written signs for the spoken symbols of words” [46, p. 115]: 5 Drimps + 1 Grimp = 16 Legs and 4 Drimps + 3 Grimps = 15 Legs. Put another way, the child described self-created images (with the help of an instrument) in the form being a precursor for the equations $5D + G = 16$ and $4D + 3G = 15$. Writing these equations using the words drimp and grimp can be mediated by the features of *Kid Pix* that may be seen as another case of double stimulation by the images of creatures as the first stimulus (Figure 4) and “speaking” (by *Kid Pix*) numerals and letters as the second stimulus (Figure 5). Whereas the picture is concrete and describes situation provided by the word problem, the equations (which are not really equations in the mathematical sense) provide foundation for seeing the picture in a more general way as the description of the manifold of situations of which the situation with drimps and grimps is just one of the many. This interplay between concrete and abstract was described by Vygotsky [45, section 3] as follows.

“Let us compare the direct image of a nine, for example, the figures on playing cards, and the number 9. The group of nine on playing cards is richer and more concrete than our concept “9,” but the concept “9” involves a number of judgments which are not in the nine on the playing card; “9” is not divisible by even numbers, is divisible by 3, is 3^2 , and the square root of 81; we connect “9” with the series of whole numbers, etc. Hence it is clear that psychologically speaking the process of concept formation resides in the discovery of the connections of the given object with a number of others, in finding the real whole. That is why a mature concept involves the whole totality of its relations, its place in the world, so to speak. “9” is a specific point in the whole theory of numbers with the possibility of infinite development and infinite combination which are always subject to a general law”.

Just as a child may not be mature enough to see in the image of the number 9 on a playing card a variety of abstractions described by Vygotsky, he or she is not yet cognitively

developed to comprehend the meaning of the word equation as a concept. Yet, the use of *Kid Pix* as an instrument of early algebra instruction makes it possible using equal sign at the level of counting and adding familiar objects within two concrete sets followed by comparing the sum of cardinalities of the two sets to a given number of objects. Paraphrasing Vygotsky, one can say that the process of formation of the concept of the system of two linear algebraic equations with two unknowns may ultimately lead one to discover that two triples of integers, (5, 1, 16) and (4, 3, 15), form a conceptual bond which, in a geometric domain, forces the straight lines described by the linear equations (informally constructed by the child within the zone of proximal development as a reflection on counting legs on drimps and grimps) to intersect at the point the coordinates of which represent the number of legs on each creature (Figures 6-8). In particular, from a problem-posing perspective, this implies that an arbitrary change of data in the conceptual bond does not necessarily allow for a solvable problem as two lines do not necessarily intersect, let alone at a point with integer coordinates. The next section will discuss the issue of problem posing from an instrumental perspective, which is especially relevant for teacher candidates' mathematics learning.

7.1.3. From problem solving to problem posing using a spreadsheet

The process of formation of the concept of two linear equations in two unknowns can motivate the construction of an instrument for posing numerically coherent (i.e., solvable) problems [5] about drimps and grimps. This instrument can be constructed in the context of a spreadsheet and, being complementary to *Kid Pix*, it could be utilized by teachers in posing a variety of problems to be solved by young children using *Kid Pix*. More generally, the instrument in question can be used to pose any kinds of problems described by a system of two linear equations in two (positive integer) unknowns. The construction of such a spreadsheet is a clear-cut TI activity (alternatively, an advanced instrumentation) for it is based on the Cramer's rule well known from linear algebra. This rule, in our case, expresses a solution to a system of two linear equations with two unknowns (the number of legs on each creature) in terms of the determinants constructed out of the above two triples of integers which serve as the coefficients of the equations. Using a spreadsheet-based instrument is a clear-cut TE activity (or instrumentalization carried out in a trial and error fashion) for it requires one to 'play' with six scroll bars and watch when two cells in the right-hand side of the instrument display a solution (the values of the unknown number of legs or just any two unknowns whatever the context). For example, as shown in Figure 5, the two triples (3, 3, 15) and (2, 3, 14) yield the solution (1, 4) – a one-legged drimp and a four-legged grimp.

The programming of the problem-posing instrument includes formulas in two cells to compute the values of unknowns through Cramer's rule (whatever the data entered into cells B4, D4, F4, B10, D10, F10 using the scroll bars) and in another two cells displaying the results of calculations in the case of positive integers only. More specifically:

Cell H2 (with hidden calculations): $= (F4*D10-F10*D4) / (B4*D10-B10*D4),$

Cell J2 (with hidden calculations): $= (B4*F10-B10*F4) / (B4*D10-B10*D4),$

Cell H6: $=IF(OR(INT(H2) < H2, INT(I2) < I2, H2 <=0, I2 <=0), " ", H2),$

Cell J6: $=IF(OR(INT(H2)<H2,INT(I2) < I2, H2<=0, I2<=0), " ", I2).$

The programming of the spreadsheet can be modified to allow for an extended context for which a non-integer solution is a possibility.

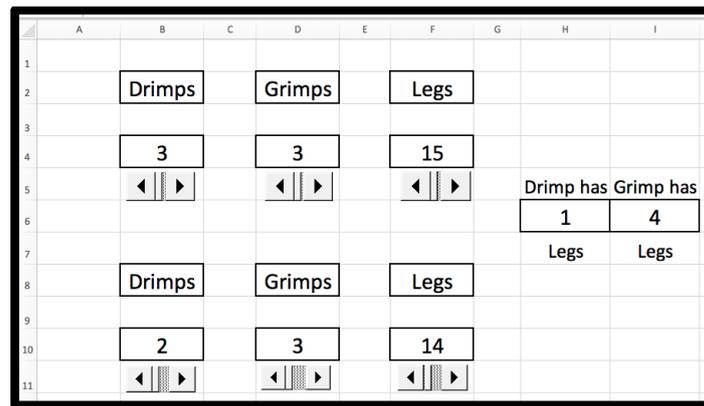


Figure 5. A problem-posing instrument.

7.1.4. On the duality of instrumentalization

As a way of extending activities from early algebra to high school algebra in the context of teacher education, the spreadsheet-based instrument of Figure 5 can be augmented by another (already mentioned) artifact, the *GC*, requiring this time only a basic level of instrumentation in constructing a graphical representation of the system of two linear equations in two unknowns and its solution displayed through placing the cursor at the point of intersection of the graphs of two equations (Figure 6). Similarly, one can use the *GSP* in creating this representation (Figure 7). However, this time, using the latter artifact as an instrument is not as straightforward as when using the former artifact. Indeed, whereas the *GC* can graph (both linear and non-linear) relations of the form $F(x, y) = constant$, the *GSP* can be used as a graphing tool when relations between two variables are presented only in the functional form, $y = f(x)$. In order to make a transition from the relation $F(x, y) = constant$ to that of $y = f(x)$, one has to go beyond the basic level of instrumentation afforded by the *GC* and make an algebraically accurate transition from one form to another in order to use the *GSP* as a graphing tool (Figure 7). Yet, in order to control the thickness of the graphs in Figure 7 one has only to select the line style from the Display menu, something that can be characterized as the basic instrumentation. Although the line style feature is not dynamic, it is still an infrastructural element of the software and, therefore, may be referred to as a “hot-spot,” a term used [24] when describing a dynamic technological environment from the instrumental genesis perspective.

At the same time, if one wants to alter the thickness of lines representing the graphs in the context of the *GC* (which can graph relations but does not have the line style tool), such alteration requires advanced instrumentation. As shown in Figure 8, in order to change the default thickness of graphs plotted by the *GC*, one has to make an accurate transition from $F(x, y) = constant$ to $y = f(x)$ and then graph the inequality $|y - f(x)| < \epsilon$ for sufficiently small ϵ – a parameter that controls the thickness. Such interplay between two similar artifacts used as instruments in carrying out the same task is interesting in a sense that the level of instrumentation required depends on which artifact has been selected. Using the above-mentioned example of instrumental genesis provided by Verillon and Rabardel [43], one can say that just as a strainer spoon and a regular spoon as artifacts can both be used in the preparation of a cup of tea, using these artifacts as instruments when dealing with leaf-loose tea requires different kind of instrumentation, depending on which of the two spoons is used. Perhaps this is what the developers of Ontario Mathematics Curriculum¹ meant by referring

¹ The university where the author works is located the United States in close proximity to Ontario province of Canada and many of the author’s students are Canadians pursuing their master’s degrees in education.

to technology as “tools of mathematicians ... [expecting that] students should be given opportunities to *select and use* the particular applications that may be helpful to them as they search for their own solutions to problems” [36, p. 15, italics added]. Such mutual duality of basic and advanced instrumentation of two seemingly similar artifacts points at the critical characteristic of the instrumental genesis – instrumentalization in the sense of technology-enabled (TE) activity may require different levels of instrumentation, on the spectrum from basic to advanced, in the sense of technology-immune (TI) activity. One can also note that advanced instrumentation is often hidden within an infrastructural feature of software that provides users with a “hot-spot” [24] tool requiring basic instrumentation only. In other words, advances in the development of artifacts can be described as instrumentalization processes due to which a transition from advanced to basic instrumentation practices becomes possible.

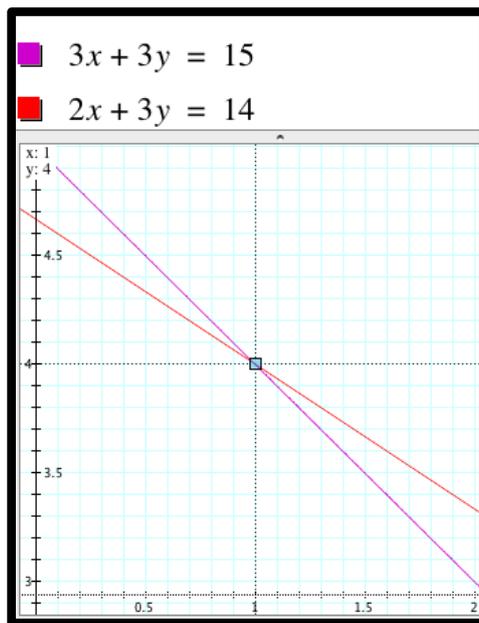


Figure 6. Graphing with the GC as basic instrumentation.

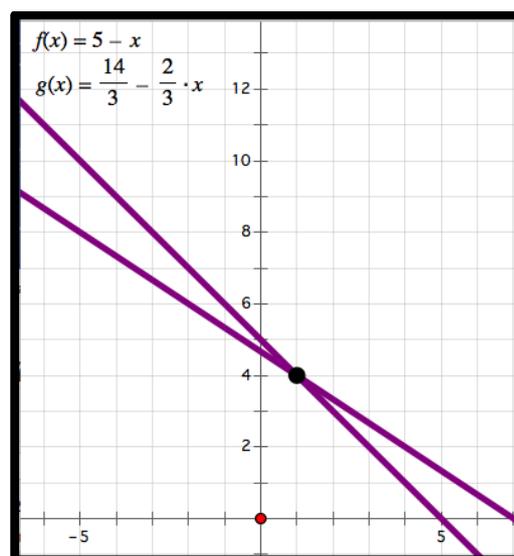


Figure 7. Graphing with the GSP as advanced instrumentation.

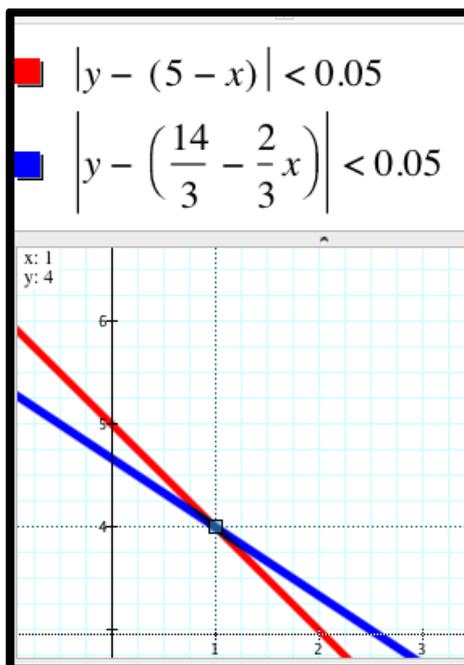


Figure 8. Changing thickness of lines with the GC as advanced instrumentation

7.2. Illustration 2: From tape diagrams to number theory with multiple instruments

As was mentioned earlier in this paper, a TITE problem cannot be solved by software at the push of a button (even when such a push is preceded by thinking about the nature of unknowns like in the case of partitioning $1/4$ into a sum of two unit fractions by WA as displayed in Figure 3), yet technology use in the process of TITE problem solving is critical. In this section, another, non-linear word problem will be explored using TITE problem-solving methodology and multiple instruments with mutually-complementary roles. Through this exploration, which will extend an earlier study of solving problems of that type in a multiple-application environment [7], the use of the instruments will be interpreted in terms of the theory of instrumental genesis.

To begin, one can check to see that the following question (a word problem)

In how many days can each of two workers complete a certain job provided that by working together they can complete the job in four days?

cannot be automatically answered (solved) by WA. In order to use this cutting-edge artifact as an instrument, one has to construct a mathematical model of the problem situation (a TI part or instrumentation) in the form of an algebraic equation and enter it into the input box of the artifact (a TE part or instrumentalization). That is, at the TI stage, WA, being inserted between the subject and the task, acts towards the former, thus supporting problem solving with epistemic mediation. To this end, one has to figure out that if x and y are unknowns representing the (integer) number of days sought and, without loss of generality, the unity is

used to represent the entire work, then the reciprocals $\frac{1}{x}$ and $\frac{1}{y}$ represent their daily

individual capacities (alternatively, average speed of their work) and the product $4\left(\frac{1}{x} + \frac{1}{y}\right)$

represents the entire work carried out over four days so that the equation $4\left(\frac{1}{x} + \frac{1}{y}\right) = 1$ or its equivalent form

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{4} \quad (1)$$

is a mathematical model of the problem using which WA can find x and y as it is shown in Figure 3. Equation (1) is a non-linear indeterminate Diophantine equation in two variables, that is, an equation with more than one integer solution. At this stage of problem solving, WA acts towards the task, thus supporting problem solving with pragmatic mediation. As a result, three possible pairs of integers representing an answer/solution to the word problem have been found: (8, 8), (6, 12) and (5, 20).

7.2.1. Solving equation (1) using tape diagrams

Another tool that can be inserted between the subject and the task is purely visual by means of which the subject can “realize the psychological operations ... necessary for the solution of the problem” [44]. More specifically, this TI part of solving the above word problem (or, put another way, to solve the indeterminate equation) is to find all the three solutions without using equation (1). Instead, one can reason with a tape diagram, a term coined in the United States by Common Core State Standards [16]. Such solution is shown in the diagram of Figure 9. To clarify, note that the top four bars show how four days of work of workers on par with each other can be divided into eight equal pieces of work, so that each worker does one piece of work daily. This means that each worker, working alone, would do the entire work in eight days and $\frac{1}{8}$ of work in one day. Thus, $\frac{1}{4} = \frac{1}{8} + \frac{1}{8}$. The four bars in the middle of the diagram show how four days of work can be divided into twelve equal pieces, so that one worker does two pieces of work daily and another (slower) worker does one piece of work daily. This means that each worker, working alone, can do the entire work in six and twelve days, i.e., $\frac{1}{6}$ and $\frac{1}{12}$ of work in one day, respectively. Thus, $\frac{1}{4} = \frac{1}{6} + \frac{1}{12}$. Finally, the four bars at the bottom of the diagram show how four days of work can be divided into twenty equal pieces, so that one worker does four pieces of work daily and another (slower) worker does one piece of work daily. This means that each worker, working alone, can do the entire work in five and twenty days, i.e., $\frac{1}{5}$ and $\frac{1}{20}$ of work in one day, respectively. Thus,

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{20}.$$

This shows that an isomorphism can be established between a TE solution generated by WA and a TI solution presented in Figure 9 through diagrammatic reasoning. In terms of instrumental genesis, reasoning with tape diagrams represents an example of double stimulation as two sets of stimuli have been involved – a three-part diagram and real-life context it represents. Nonetheless, one might notice that the first set of stimuli did have bars divided into two, three and five pieces, and ask as to why dividing a bar into four pieces was not included. This suggests that establishing an isomorphism between two solutions, the TE and the TI ones, can only validate the accuracy of the solutions but it does not prove that all solutions to equation (1) have been found.

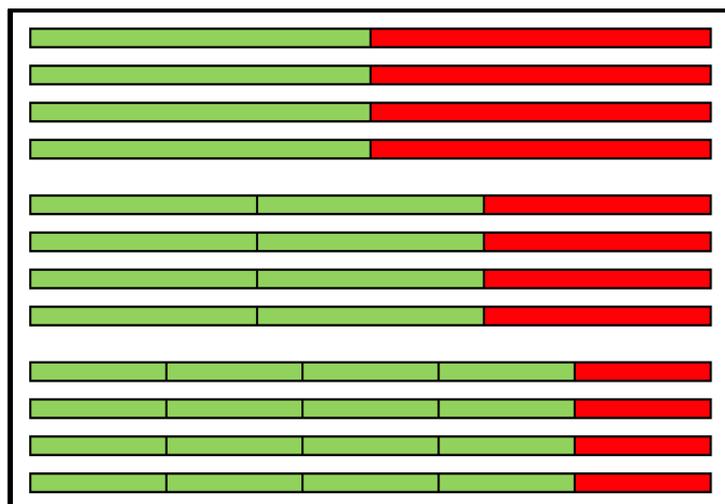


Figure 9. Solving work problem using diagrammatic reasoning.

7.2.2. Algebra as a psychological instrument of proof

Another TI part of this activity could be in making sense of the three solutions to equation (1) generated by WA (or found through reasoning with tape diagrams). To this end, the following question has to be addressed: How can one prove that equation (1) has indeed only three solutions? There are several ways to answer this question. One way is to introduce

the identities $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$ and $\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n}$ to imply that when $n = 4$ the only

candidates for a solution beyond those generated by the two identities, that is, $\frac{1}{4} = \frac{1}{5} + \frac{1}{20}$ and

$\frac{1}{4} = \frac{1}{8} + \frac{1}{8}$, are the fractions $\frac{1}{6}$ and $\frac{1}{7}$; the latter one would have to be rejected, as the

difference $\frac{1}{4} - \frac{1}{7} = \frac{3}{28}$ is not a unit fraction. This exploration can also lead to posing new problems by altering the right-hand side of equation (1). Through this kind of problem posing, TI and TE activities become indistinguishable and converge into a single tool-mediated mathematical activity. Problem posing can also be seen through the lenses of instrumental genesis – a new pedagogical idea about using problem-solving data as “a means by means of which we direct and realize psychological operations” [44] for posing new problems, thus enriching the existing curriculum. In connection with the observation made at the end of the previous section, one can note that whereas algebra-based proof did provide rigor, something that the tape diagram approach did not, this observation has yet to be explained.

7.2.3. Geometry software as an instrument of proof

One can also demonstrate that equation (1) has exactly three solutions by representing unit fractions through electronic fraction circles constructed in the context of the *GSP*. In many cases, this computer program can serve as a friendly, ready-to-be-used instrument. As was mentioned above, constructing a point with given coordinates, plotting graphs of functions, and changing the line style of the graphs are examples of the basic instrumentation. However, in the context of proving that equation (1) has exactly three solutions, a special instrument has to be created to enable pragmatic mediation of the proof to be carried out in a geometric domain. To this end, a mini-program called a script has to be developed at an

advanced level of instrumentation. Typically, scripts are used for specific, multi-step geometric constructions. Consider a task of constructing and then adding two fraction circles (unit fractions) in order to make a larger fraction. In the fractional form, the task was solved above “at the push of a button” by using WA (Figure 3) when representing $1/4$ as a sum of two unit fractions. In a course of construction of a fraction circle one has to define the location of its center and the length of its radius. When constructing the sum of two (or more) unit fractions, their representations through fraction circles (or simply sectors of the circle) should have common center and the same length radii.

The main instrumentation (or TI) idea behind this construction task deals with a frequently overlooked (or taken for granted) fact that fractions may only be added when the same unit is their point of reference. This idea is hidden in the construction of a fraction circle when one defines its radius. As shown in Figure 10, the three pairs of fraction circles have the same radius, each pair sharing the center. Therefore, they may be added as geometric images, something that is taken for granted when the corresponding unit fractions are added in the numeric domain. Here, the concept of same unit is implicitly embedded in the construction of the very unit using its parts. A conceptual flaw which might occur at the action level (constructing fraction circles with different radii) might result in the erroneous image of the sum of such fractions. The three solutions follow from representing fraction circle $1/4$ first as a sum of two equal fractions ($1/8$ and $1/8$) leaving for unequal fractions only three options: one with $1/7$ (bigger than $1/8$), another with $1/6$ (bigger than $1/7$), and the last one with $1/5$ (bigger than $1/6$). While this geometric approach is similar to the one described above in terms of algebraic identities, it provides one with a visually rich instrumental genesis approach to proof. Furthermore, one can see that dividing a bar in the tape diagram into four pieces is equivalent to dividing the fraction circle $1/4$ into two pieces one of which would be represented by the unit fraction $1/16$ and, due to the equality $\frac{1}{4} - \frac{1}{16} = \frac{3}{16}$, the remaining piece would be represented by the fraction $\frac{3}{16}$ which is not a unit fraction. Finally, due to the use of geometric reasoning as a means of proof the process of instrumentation does take place as one grows in conceptual understanding of ideas behind the arithmetic of fractions.

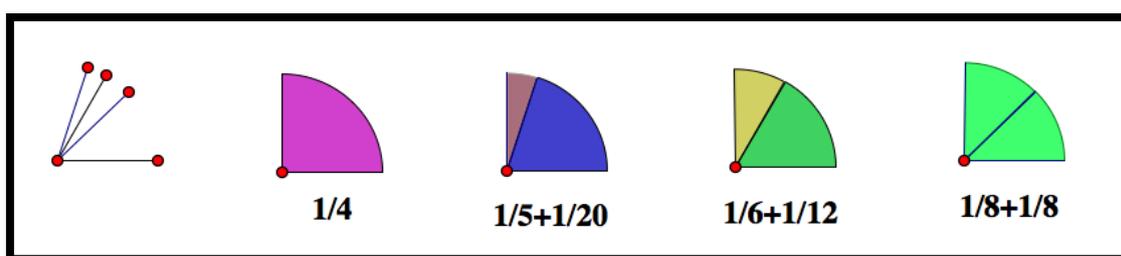


Figure 10. Pragmatic mediation by a GSP-based instrument.

7.2.4. Number theory as a psychological instrument of generalization

The above two ways of proving the number of solutions of equation (1) were based on ideas dealing with algebraic identities and geometric representations of operations on fractions. A more advanced proof may deal with the subject’s use of number theory. To this end, one has to represent equation (1) in the form

$$y = 4 + \frac{16}{x - 4} \tag{2}$$

(where, not by coincidence, $16 = 4^2$) and show that only when the denominator $x - 4$ assumes the values 1 (when $x = 5$ thus $y = 20$), 2 (when $x = 6$ thus $y = 12$), 4 (when $x = 8$ thus $y = 8$), 8 (when $x = 12$ thus $y = 6$), and 16 (when $x = 20$ thus $y = 5$), equation (2) has five whole number solutions two of which are symmetrical, thus leaving us again with exactly three solutions. One can note that each solution with unequal x and y has a symmetrical counterpart and therefore, in order to exclude symmetrical solutions from being counted twice, one can increase five by one and divide the sum by two. Equations (1) and (2) can be generalized to

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

$$y = n + \frac{n^2}{x - n}, \tag{3}$$

respectively, so that the number of partitions of the unit fraction $1/n$ into a sum of two like fractions would be equal to one half of the (odd) number of divisors of n^2 increased by one. Note that an integer square has always an odd number of divisors. Indeed, if $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ (the prime factorization) then $n^2 = p_1^{2r_1} p_2^{2r_2} \dots p_k^{2r_k}$ and thereby, by the rule of product², there are $(2r_1 + 1) \times (2r_2 + 1) \dots (2r_k + 1)$ divisors of n^2 as each prime factor p_i can be selected in $2r_i + 1$ ways with exponents ranging from zero to $2r_i$.

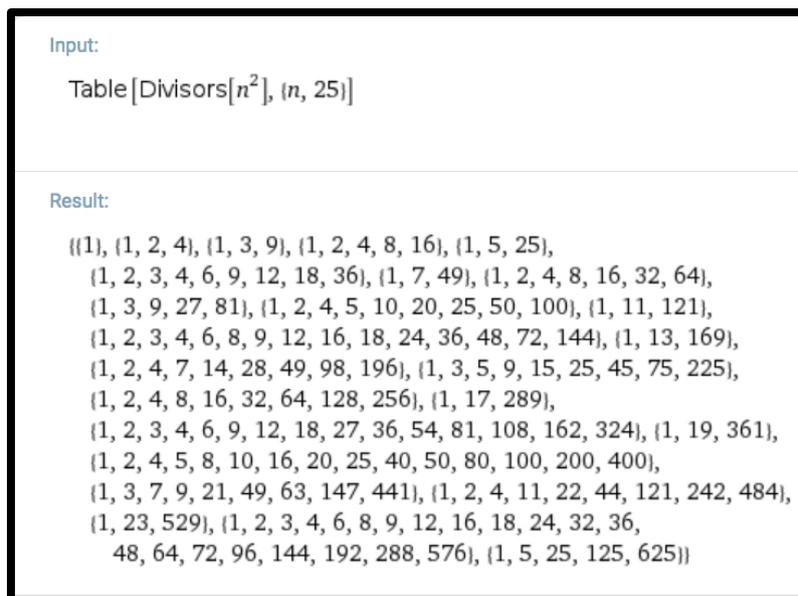


Figure 11. WA shows that an integer square has an odd number of divisors.

This reasoning can be supported by WA which, can demonstrate that square of any integer has an odd number of divisors (e.g., in Figure 11 all divisors of the first 25 squared

² The rule of product means the following: if object A can be selected in m ways and if, following the selection of A, object B can be selected in n ways, then the ordered pair (A, B) of the two objects can be selected in mn ways. For example, if Orrin wants to buy a donut and a soft drink and there are three kinds of donut and four kinds of drink, then there are $3 \times 4 = 12$ possible choices of donut and soft drink that he can make.

integers are displayed in brackets). Therefore, there are $\frac{(2r_1 + 1) \times (2r_2 + 1) \dots (2r_k + 1) + 1}{2}$ ways to partition $1/n$ into a sum of two unit fractions. When $n = 4$ we have $n^2 = 2^4$, that is $r_1 = 2, r_i = 0, i > 1$ and $\frac{(2r_1 + 1) + 1}{2} = \frac{5 + 1}{2} = 3$ – the number of ways the unit fraction $1/4$ can be partitioned into a sum of two like fractions. In that way, WA becomes inserted between the subject and the task of proving and it acts bi-directionally in supporting the use of number theory as a psychological instrument of generalization.

7.3. Illustration 3: Mediating experimental mathematics by multiple instruments

In this final illustration, another type of equation, a two parametric difference equation commonly used nowadays as a model of a discrete dynamical system, will be explored. In what follows, a TITE methodology will be presented as an educational analogue of experimental mathematics and described in terms of the theory of instrumental genesis.

7.3.1. Pragmatic mediation using a spreadsheet

To begin, consider a mathematical model in the form of a linear difference equation of the second order

$$x_{n+1} = ax_n + bx_{n-1}, x_0 = x_1 = 1. \tag{4}$$

Using a spreadsheet as an artifact, that is, a tool to be turned into an instrument through a relatively uncomplicated TI programming activity, the subject can set a goal to model equation (4) numerically and, similar to the case of Fibonacci number sequence described by this equation when $a = b = 1$, explore the behavior of the ratios $\frac{x_{n+1}}{x_n}$ for different values of a and b (Figure 12).

This requires the following (basic instrumentation) skills in spreadsheet programming:

- 1) designating two (perhaps slider-controlled) cells for the values of a and b (both positive and negative),
- 2) setting the initial values in two successive cells, and
- 3) defining in the next cell a formula through which, according to equation (4), a linear combination of the contents of the previous two cells is created.

As a result, the spreadsheet displays the following three values needed to solve equation (4) numerically: $x_0, x_1, ax_1 + bx_0$. Now, one has to define another formula, to the right of the cell with x_1 to compute the ratio $\frac{x_1}{x_0}$. It only remains to replicate the so-developed formulas down

the rows (a TE part of the instrumental genesis) and, by changing the values of a and b , to observe the results of this computational experiment. These uncomplicated TITE activities turn an electronic spreadsheet into an instrument of experimental mathematics, enabling pragmatic mediation of the experiment in question.

Coincidentally, one can set $a = 2$ and $b = -4$ (Figure 12) noting that $a^2 + 4b < 0$; this inequality implies that the characteristic equation, $x^2 - ax - b = 0$, of difference equation (4) does not have real roots. As a result, one can observe that the ratios $\frac{x_{n+1}}{x_n}$ form cycles of period three represented by the string of numbers $\{1, -2, 4\}$ repeated over and over as n increases. Computing (or unintentionally discovering through ‘playing’ with the coefficients

a and *b*) this three-cycle is a technology enabled (TE) part of the mathematical activity pragmatically mediated by the spreadsheet. However, an unexpected experimental result calls for explanation. In particular, a subject might wonder: Is there a rule that can produce coefficients *a* and *b* so that the ratios $\frac{x_{n+1}}{x_n}$ always form cycles of period three? Answering this question is not possible without recourse to mathematical reasoning. At that point, another TI part of the activity begins, and it may require the use of tools capable of symbolic computations and enabling epistemic mediation of the subject.

<i>a</i>	x_n	x_{n+1}/x_n
2	1	
◀ █ ▶	1	1
	-2	-2
	-8	4
	-8	1
	16	-2
	64	4
<i>b</i>	64	1
-4	-128	-2
◀ █ ▶	-512	4
	-512	1
	1024	-2
	4096	4
	4096	1
	-8192	-2
	-32768	4

Figure 12. Turning an artifact into an instrument.

7.3.2. WA and Maple as instruments of epistemic mediation

In order to explore the behavior of the ratios $\frac{x_{n+1}}{x_n}$, difference equation (4) can be transformed to the form

$$r_n = a + \frac{b}{r_{n-1}}, r_0 = 1 \tag{5}$$

where $r_n = \frac{x_{n+1}}{x_n}$. It was observed that when $a = 2$ and $b = -4$, equation (5) generates a three-cycle $\{1, -2, 4\}$. Therefore, in order to provide an explanation to this unexpected phenomenon, the subject has to deal with equation (5) keeping in mind that $a^2 + 4b < 0$. The questions to be answered are:

Can equation (5) generate a three-cycle when $a \neq 2$ and $b \neq -4$?

Can equation (5) generate cycles of higher lengths for certain values of parameters *a* and *b* such that $a^2 + 4b < 0$?

What are those values and how does a cycle's length depend on them?

To answer these questions, a new series of TITE activities ensues.

In order to find all cycles of period three, the subject can use equation (5) to compute (without using technology)

$$r_1 = a + b, r_2 = a + \frac{b}{g_2} = a + \frac{b}{a + b} = \frac{a^2 + ab + b}{a + b},$$

$$r_3 = a + \frac{b}{g_3} = a + \frac{b(a + b)}{a^2 + ab + b} = \frac{a^3 + a^2b + 2ab + b^2}{a^2 + ab + b},$$

and then, after setting $r_3 = 1$, to solve the equation $\frac{a^3 + a^2b + 2ab + b^2}{a^2 + ab + b} = 1$. This would lead to the following chain of equalities

$$a^3 + a^2b + 2ab = a^2 + ab + b, \quad a^3 + a^2b - a^2 + ab - b = 0$$

$$a^2(a + b - 1) + b(a + b - 1) = 0, \quad (a + b - 1)(a^2 + b) = 0$$

whence either $a + b - 1 = 0$ or $a^2 + b = 0$. Alternatively, replacing the above TI activity with a TE one, the subject can use WA as a tool capable of symbolic computations.

Note that when $b = 1 - a$ equation (5) generates $r_n = 1$ for all $n \geq 0$. Indeed, if $b = 1 - a$ in (5) then $r_n = a + \frac{1-a}{r_{n-1}}$ whence $r_0 = r_1 = r_2 = \dots = 1$, something that may be considered as a trivial cycle of any period, in particular, of period three. Therefore, only those parameters a and b that satisfy the equation $a^2 + b = 0$, provide equation (5) with a non-trivial cycle of period three. Put another way, in the (a, b) -plane all three-cycles reside on the parabola defined by the equation $a^2 + b = 0$ which, in particular, passes through the point $(2, -4)$.

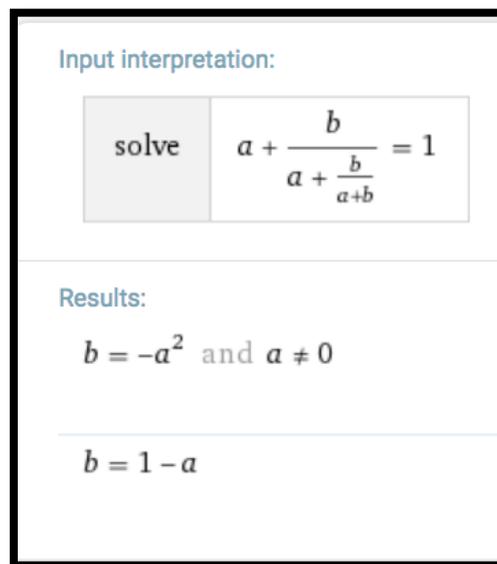


Figure 13. Taking advantage of symbolic computations of WA.

As shown in Figure 13, WA, used as an instrument, confirms the correctness of the traditional algebraic transformations. It is always useful to have formal algebraic transformations confirmed by an instrument and vice versa, to have symbolic computations delivered by the instrument to be confirmed by algebra just to make sure that correct button on the artifact was pushed. Such dual confirmation would motivate the subject to use the instrument in the case of more complicated symbolic computations. This is shown in Figure 14 where WA found that the relation $a^2 + 2b = 0$ enables the emergence of cycles of period

four in recursive equation (5). Note that when $b = -a^2$ and $b = \frac{-a^2}{2}$ the inequality $a^2 + 4b < 0$ holds true; put another way, the parabolas $b = -a^2$ and $b = \frac{-a^2}{2}$ are located under the umbrella of the discriminant parabola $b = \frac{-a^2}{4}$. An interesting aspect of these explorations is the emergence of more than one parabola as a locus of parameters a and b which cause the ratios $\frac{x_{n+1}}{x_n}$ in equation (4) to form cycles of any period greater than four. For more information see [6].

Input interpretation:

solve	$a + \frac{b}{a + \frac{b}{a+b}} = 1$
-------	---------------------------------------

Results:

$a = 0$ and $b \neq 0$

$b = -\frac{a^2}{2}$ and $a \neq 0$

$b = 1 - a$

Figure 14. Computing a four-cycle using WA.

$r1 := 1$	$r1 := 1$
$b := -\frac{a^2}{2}$	$b := -\frac{a^2}{2}$
$r2 := a + b$	$r2 := a - \frac{1}{2} a^2$
$r3 := \text{simplify}\left(a + \frac{b}{r2}\right)$	$r3 := \frac{a(-1+a)}{-2+a}$
$r4 := \text{simplify}\left(a + \frac{b}{r3}\right)$	$r4 := \frac{a^2}{-2+2a}$
$r5 := \text{simplify}\left(a + \frac{b}{r4}\right)$	$r5 := 1$

Figure 15. Verifying locus of four-cycle using Maple.

7.3.3. Verifying theory through experiment using complementary instruments

Now, as a way of verifying theory, stemming from symbolic computations of *WA* and *Maple*, through a new experiment carried out by the joint use of the *GC* and a spreadsheet as complementary instruments, the subject can graph the parabola $b = \frac{-a^2}{2}$ (Figure 16 where $b = y$ and $a = x$) by the *GC*, set up a multi-digit precision in the Document Setting of the program, select a point on the parabola using the cursor, and then enter manually the coordinates of the selected point into the spreadsheet (Figure 17 where $a = x = 1.765625$ and $b = y = -1.55871582$) to see the four-cycle $\{1, 0.206909, -5.76771, 2.035874\}$. In much the same vein, the subject can continue using *WA* or *Maple* to symbolically compute the equations of loci on which cycles of higher periods occur and verify the computations through the use of the *GC* and a spreadsheet as complementary instruments. As noted by Freudenthal, “It is independency of new experiments that enhances credibility ... [for] repeating does not create new evidence, which in fact is successfully aspired to by independent experiments” [21, pp. 193-194].

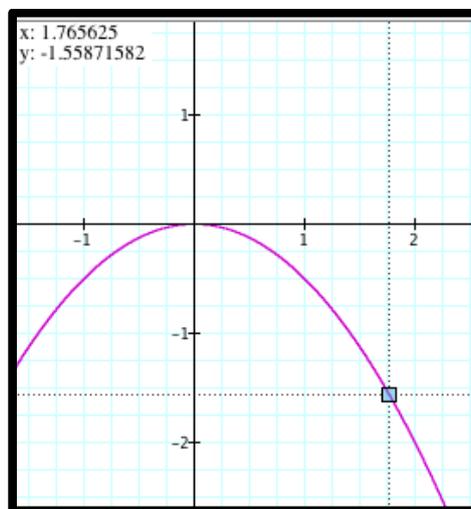


Figure 16. The *GC* as a complementary instrument to *WA*.

a	x_n	x_{n+1}/x_n
1.765625	1	
◀ █ ▶	1	1
	0.206909	0.206909
	-1.19339	-5.76771
	-2.4296	2.035874
	-2.4296	1
	-0.50271	0.206909
b	2.899459	-5.76771
-1.55871582	5.902932	2.035874
◀ █ ▶	5.902932	1
	1.221371	0.206909
	-7.04451	-5.76771
	-14.3417	2.035874

Figure 17. Complementary use of a spreadsheet to the *GC* and *WA*.

The joint use of the *GC* and the spreadsheet demonstrates how the subject's utilization of multiple mutually-complementary instruments depends on a specific practice. When graphing a parabola, the form in which its equation is entered into the input box of the *GC* is irrelevant for the instrument. It may be both a relational form such as $a^2 + 2b = 0$ and functional form such as $b = \frac{-a^2}{2}$. In both cases, same geometric image, a graph, which has to be used jointly with numeric characteristics, results. Yet, the precision with which the coordinates of a point through which the graph passes are picked up by the cursor from the graph is essential for the spreadsheet, a complementary instrument, which uses the coordinates as an input for the instrumental activity in order to a computational experiment to be accurate. This accuracy is especially important when the goal of a new experiment is to enhance credibility of earlier experiments. Although the *GC* and the spreadsheet are independent instruments, their complementary character in the instrumental activity described above demands that "their specificities differentiate their functional values in line with the particularities of situations" [38, p. 679].

Experimentally more difficult, although not impossible, the verification of complementariness of the spreadsheet and the *GC* can include using scroll bars which control the values of the parameters a and b in the spreadsheet to obtain a cycle of period four, then graph the so located values of a and b in the context of the *GC* to see that the point belongs to the parabola $a^2 + 2b = 0$. By zooming in, the subject can decide whether precision with which the values of the parameters have been found is of sufficient accuracy for the numeric, geometric, and algebraic outcomes of the instrumental activity in order to be experimentally close to each other. If they are, one can say that the TITE methodology in which different instruments and instrumental activities overlap, has the potential to transform educational practices of mathematics teacher preparation. The next section includes teacher candidates' reflections on the use of multiple mutually-complementary instruments as they learn teaching mathematics in the digital era.

8. Voices from teacher education classrooms

Rabardel and Bourmaud [38] proposed to distinguish among three major orientations of an instrument-mediated activity when an instrument is placed between the subject (e.g., a user of digital tools) and the object (e.g., a mathematical problem). These include the orientation toward the object (a pragmatic mediation aimed at solving a problem), the orientation toward oneself (an epistemic mediation in the case when problem solving becomes a means of the subject's intellectual growth), and the orientation toward other subjects (i.e., epistemic mediation of other possible actors of the instrument-mediated activity). The third orientation has important implications for mathematics teacher education when its course work integrates TITE methodology and the use of multiple mutually-complementary instruments. In the education of teachers, other actors are either students in K-12 schools where teacher candidates do field work or their future students for whom the impact of the third orientation is delayed. In real life, there are many technical and social systems in which delays are inherent. One such system is education where application of knowledge received by a teacher candidate (the original subject) is delayed for some time. Eventually, this orientation of the subject toward other subjects may be associated with both pragmatic mediation, when a teacher's task is to teach, and epistemic mediation, when a teacher learns teaching by interacting with students.

The following comments by teacher candidates about their use of digital instruments have been solicited over the span of several years as reflections on different courses taught by the author under the umbrella of TITE methodology. They are included to demonstrate

orientation of teacher candidates as subjects towards their future students as other subjects. The comments are related to the above three illustrations of TITE problem solving as instrumental genesis using multiple mutually-complementary instruments.

8.1. Reflections on Illustration 1

Reflecting on the use of *Kid Pix* as a tool for early algebra learning, an elementary teacher candidate shared her perspective on learning to use computers and ultimate plans to utilize the skills so developed in the future. She believes that *“Creating visuals on the computer is a great way for students to learn how to use technology for school purposes and to utilize the many skills students have. I will definitely be using these new skills in my classroom when I become a teacher. I think it is a great resource to have when teaching elementary school students as they learn best by using visuals.”*

In this comment, one can see the orientation of the teacher candidate (a subject) toward other subjects (young children) delayed until her dream to become their teacher comes through. In the context of the same illustration, another teacher candidate's comment is appreciative of turning a spreadsheet as an artifact into an instrument for posing problems. As she put it, *“Technology makes problem posing a lot easier for teachers. Once the spreadsheet is developed, it is very easy to go back and change the numbers. It can also help teachers save paper, by projecting problems onto a smart board, instead of printing out handouts for each of their students.”*

Note that the teacher candidate, by using the word “developed” distinguished between a spreadsheet as an artifact and an instrument, and opened a window to seeing such development as a TI activity or instrumentation. In her comment, a smart board appears as a complementary instrument of classroom pedagogy which works in combination with any other tool capable of projecting images on a screen through a TE activity or instrumentalization.

8.2. Reflections on Illustration 2

The second illustration included the use of the *GSP* as a means of proving that the fraction $\frac{1}{4}$ has exactly three partitions into a sum of two unit fractions. As is well known, the arithmetic of fractions is a difficult topic to teach in the grade school and it is not always well accepted by elementary teacher candidates as they take an elementary mathematics education course due to their previous confusion with this topic. That is why, when the use of technology is considered by teacher candidates as help in their preparation to become professionally oriented toward other subjects, their own students included, one can recognize the value of instrumentation in turning a geometric software into an instrument for teaching and learning fractions. The following comment by a teacher candidate is indicative in that respect. When asked to reflect on the issue of mediating mathematical proof by means of technology, a teacher candidate admitted: *“The electronic manipulatives helped me articulate mathematical proof through visual understanding. Sometimes it is difficult to consider which fractions are going to create a particular sum. But the manipulatives helped me to understand what was going on in the problem as well as understand why the answers were correct. This is my first experience with the Geometer's Sketchpad and I was thrilled at knowing there are alternative ways to teach children about fractions. I can remember disliking the topic in class when I was in school because it was boring and often difficult to understand.”*

In this comment, one can see the emergence of the teacher candidate's professional orientation toward her future students. Although, like in the case of the problem-posing

spreadsheet of Figure 5, a TI or instrumentation part was due to the author, the teacher candidate was capable of carrying out a TE part or instrumentalization. The same can be said about a comment of another teacher candidate who clearly saw the *GSP* as a psychological tool being inserted between the subject and the object. *“The use of electronic manipulatives helped in two ways. It helped by backing up the equations by showing that there was no way that any unit fractions were excluded, and it showed how the problem could be solved by using pictures only. When it comes solving math problems, you always want to be sure that you have found every answer possible and that nothing was left out. If it was solved mathematically first, then the manipulatives proved your answer once you finished solving the problem.”* Here, the teacher candidate essentially talked about the value of double stimulation of mathematical reasoning by means of formal mathematics and instrumental action as “a means by means of which we direct and realize the psychological operations (memorizing, comparing, selecting, etc.) necessary for the solution of the problem” [44].

8.3. Reflections on Illustration 3

Finally, in the context of the third illustration aimed at the demonstration of the TITE-oriented educational side of experimental mathematics, the following two comments are indicative of secondary teacher candidates’ appreciation of the affordances of technology in using experiment as window to theory and verifying theory so developed through a new experiment. The appropriate use of multiple digital instruments has not come unnoticed in these comments. So, one of the future secondary mathematics teachers admitted: *“Technology is very helpful in developing a relationship between experimental and theoretical mathematics. Experimental mathematics can be achieved through uses of technology such as spreadsheets, graphing calculators, and Maple which would have been inefficient if attempted algebraically and without the use of technology. Technology can also aid students in theoretical aspects of mathematics when algebra and operations become too tedious or difficult.”* One can also see in this comment the acknowledgement of the value of digital instruments in facilitating access of *all* students to advanced symbolic mathematics, a long-term program initiated with the advent of technology into K-16 mathematics classroom at the end of the last century [27, 28, 41, 42]. Also, complementary use of instruments requires understanding of their mutual dependence in terms of the importance of mathematical precision of the input and its influence on the output. Indeed, as a teacher candidate acknowledged, *“Using points traced on the parabola and inputting them into a spreadsheet yield a fascinating observation as the ratios form a cycle. Using technology to visualize this phenomenon illustrates how sensitive the software is. If the accuracy of decimal places is less than three decimal places, the ratios do not form a cycle.”* One can see the teacher candidate’s attention to the conditions of the experiment and the appropriate use of an instrument as an element of the second set of stimuli used in verifying the validity of instrumental actions for mastering the first set of stimuli – a TITE-based computational experiment aimed at exploring the behavior of solutions of recursive equation (5) under the condition $a^2 + 4b < 0$.

9. Conclusion

This paper was written to demonstrate connections between the theory of instrumental genesis and TITE mathematical problem-solving methodological framework. Whereas teaching and learning through the TITE framework was developed in the context of mathematics education, the theory of instrumental genesis studies pedagogical ideas about the uses of digital technology across the wide spectrum of disciplines and psychological contexts.

However, when mapped on the domain of mathematics education, the theory of instrumental genesis has many aspects in common with the TITE mathematical problem solving. The paper provided three major illustrations to discuss connections between two perspectives on the use of digital instruments in mathematics education. The focus of discussion was on the use of multiple instruments oriented toward a single task. At the same time, the use of multiple instruments made it possible to consider a task as an evolving exploration because each instrument, being both complementary and unique, allowed for a specific extension of the task. This also made it possible to extend the three major orientations of an instrument-mediated activity proposed in [38] – toward the object, toward oneself, and toward other subjects – to include an orientation toward other objects. In mathematics education, other objects could be construed as new problems posed by the subject through the process of epistemic mediation as new instrumentation ideas develop.

The diagram of Figure 18 shows multiple orientations motivated by bi-directional relations among the instruments involved, among the subjects, and among the objects stemming from instrument-mediated activities. For example, the interaction among four instruments was illustrated through exploring a non-linear recursive equation (5) where the cyclic behavior of solutions was first discovered numerically within a spreadsheet, then explored in a symbolic environment of *WA* and *Maple*, and finally verified through the joint use of the *GC* and a spreadsheet in order to enhance credibility of instrument-mediated activities responsible for the serendipitous numeric discovery and rigor-laden algebraic realization of cycles. The orientation toward other objects is a natural characteristic of such learning environments because problem solving and problem posing are closely related mathematical activities [20, 29, 40]. As for the issue of orientation toward other subjects, solicited responses of teacher candidates on a TITE course work involving multiple instruments used in the context of three different illustrations were presented and briefly discussed. A few more comments about these illustrations are also included in the conclusion.

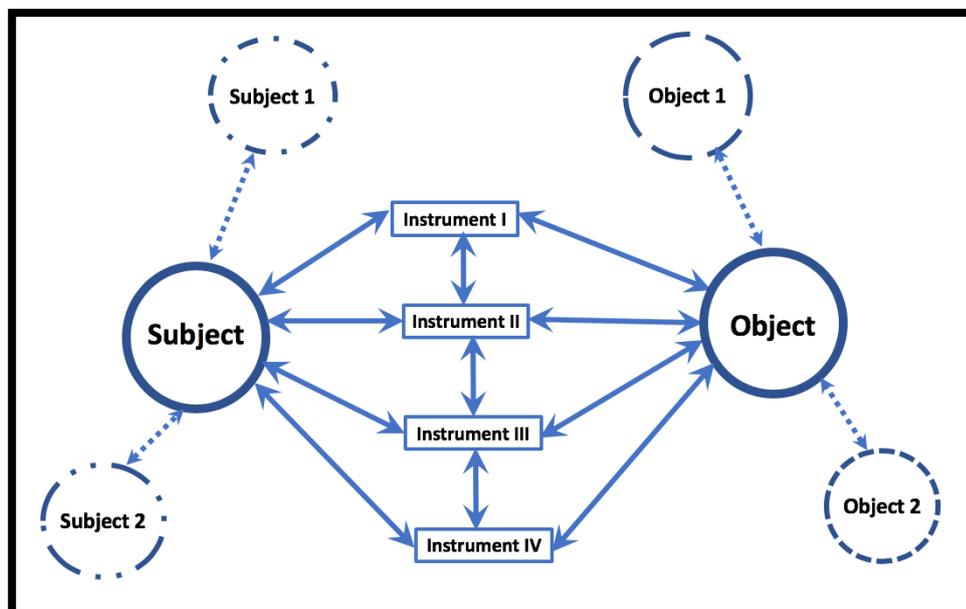


Figure 18. TITE activity with mutually-complementary instruments and multiple orientations.

The first illustration involved multiple instruments used within a single context, yet at different grade levels associated with the context. It demonstrated the progression of the use of instruments from drawing artificial creatures by young children to posing similar problems and constructing graphical solutions of the problems from early algebra curriculum by their future teachers. It was noted that through posing problems and exploring them using multiple instruments, one discovers that problem data complies a conceptual bond the appreciation of which is what unites problem posing and problem solving. The approach to early algebra when the appropriate use of software for creative activities of young children, under the guidance of a teacher as a ‘more knowledgeable other’, allows them to solve informally a system of two linear equations in two unknowns as the first step toward learning algebra at the secondary level. That is, algebraic skills that are typically developed at the secondary level can, nonetheless, be observed in a young learner, when he or she is appropriately guided by a teacher, without the former realizing the possession of such skills as they are at the rudimentary stage of development. Vygotsky described this complex educational psychology phenomenon as follows (*cf.* the last sentence of the first paragraph of section 5 of this paper):

“Although at an early stage of mathematical development, quantitative reasoning and arithmetical thinking of a child are pretty vague and immature in comparison with those of an adult with whom the child interacts, it is through this interaction that the final forms of reasoning and thinking about numbers, that have to be developed as a result of having an adult in his/her environment, are somehow present at that stage and, not only present but in fact define and guide the child’s first steps toward the development of the final forms of understanding quantity and comprehending arithmetic” [48, p. 84, translated from Russian by the author].

The second illustration demonstrated that a tape diagram had the same functional value as algebraic, geometric and number theory reasoning tools (or psychological operations) in terms of the goal which was to establish an isomorphism among the sets of three solutions of non-linear algebraic equation (1) found through different means. Yet, “their specificities differentiate their functional values in line with the particularities of situations” [38, p. 679] enabling the demonstration that different concepts of mathematics represent a system of psychological instruments that are not isolated but rather can merge by and large on a single task ensuring the clarity of its completion through conceptual connectivity. At the same time, only algebra, geometry, and number theory provided rigor in proving that equation (1) has exactly three solutions. It was also argued that rigor of algebra (or number theory, for the matter) may not be used to explain some simple observations made in the context of the tape diagram. Yet, geometric reasoning with the use of fraction circles was helpful in explaining those observations. This suggests that the tape diagram and the fraction circles approaches were close to each other in a sense that they both belong to the psychological domain of imagery and visualization, something that is critical for Gestalt psychologists who study constructive aspects of perception in terms of what one can see in an image (e.g., [50]).

The third illustration dealt with exploring recursive equation (5) with a variety of instruments including a spreadsheet, *WA*, *Maple*, and the *GC*. It was demonstrated that whereas *WA* and *Maple* as different artifacts had the functional complementariness in obtaining and confirming the equations of parabolas as loci on which cycles of the same period realize³, the graphic and numeric functionalities of the *GC* and a spreadsheet, respectively, not only confirmed the correctness of symbolic computations but demonstrated

³ In this paper such demonstration was limited to period four. As shown in [6], for any integer $K > 0$ there exists an integer $p > K$ such that in equation (4) the ratios x_{n+1} / x_n form cycles of period p .

how the appropriate use of the system of four diverse instruments “organized resources of a heterogeneous nature into a homogeneous system whole” [38, p. 682]. The “system whole” made it possible to demonstrate an educational side of experimental mathematics and its capacity to provide both internal and external validity of an educational experiment [13] – providing meaningful interpretation of the former type of validation (recognizing the possibility of cyclic behavior of the ratios x_{n+1} / x_n) and using this interpretation to generalize the experiment (with support of its instruments capable of symbolic computations) as the latter type of validation. The “system whole” can also show the progress in the use of technology in mathematics education over the span of several decades from noting that “calculus should be presented to the student in the same spirit as the experimental sciences” [9, p. 664] to arguing that the pedagogy of the digital-era school mathematics “can be used to *experiment* with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave” [16, p. 62, italics added].

Digital tools considered in this paper can often be observed as software of choice in a mathematics teacher education classroom. In the case of a spreadsheet, the transition from an artifact to an instrument requires basic skills in using cell referencing techniques and spreadsheet formulas. In some cases, the creation of a spreadsheet environment requires rather advanced knowledge of Excel or its appropriate integration with tools of symbolic computation. In the case of *WA* or *Maple*, the main difficulty in their instrumentation deals with conceptual understanding of the task to be solved by the tool. Such understanding required knowledge of the ranges of unknowns for *WA* and relationships between parameters for *Maple*. One can say that in the joint use of *WA* and *Maple* “there is both a partial functional coverage (redundancy) and equally partial functional complementarities” [38, p. 682]. In the case of *KidPix*, the transition from artifact to instrument requires understanding of how actions on the first-order symbols which comprise the artifact can be interpreted through the second-order symbolism of a specific early algebra task. Moreover, the suggested use of *KidPix* as an artifact, designed originally for kids’ entertainment, is a clear case of instrumentation when a non-disciplined creativity of children is advanced to be used within the educational domain of early algebra. Likewise, through learning to replace high school algebra by conceptually-oriented arithmetic teacher candidates grow intellectually and use their instrumentation skills for the creation of a spreadsheet-based (alternatively, artifact-based) computational environment as a self-made instrument which can be used for mathematical problem posing. Through posing problems as an instrumental activity stemming from problem solving one can come across “problems being defined by answers at the same time as answers are being constructed during the shaping of problems” [30, p. 65]. Constructing such instruments for formulating (and solving) mathematical problems, that can be utilized beyond a task that motivated the very idea of the construction, contributes to the pragmatic dimension of artifacts integrated into the praxis of technology-rich classrooms [43].

Throughout the paper, the difference between basic and advanced instrumentations was discussed to demonstrate how the level of instrumentation required depends on an artifact. For example, whereas the change of the line style (e.g., thickness) by a “hot-spot” of the *GSP* requires basic instrumentation, the change of thickness in the context of the *GC* required advanced instrumentation through digital fabrication. Furthermore, the level of instrumentation, in many respects, depends on a subject as well: what seems to be advanced for one may be basic for the other. In that sense, the orientation toward other subjects can be used as an instrument of professional development of schoolteachers within a school or a school district. The array of artifacts available and the scope of their applications in a mathematics teacher education classroom support the diversity of teaching methods incorporating the ideas of instrumental genesis and TITE mathematical problem solving. In that way, the instrumental perspective on TITE mathematics curriculum of teacher education

opens a window to new uses of commonly available digital tools to the benefit of diverse learners of K-16 mathematics.

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