IMVI *ISSN (p) 2303-4882, ISSN 1840-4383*
Open Mathematical Education Notes
Vol. 10(2020), 33-43
www.imvibl.org / JOURNALS / IMVI OMEN
DOI: 10.7251/OMEN2002033A

PIZZAS, EGYPTIAN FRACTIONS AND PRACTICE-FOSTERED MATHEMATICS

Sergei Abramovich

State University of New York, Potsdam, United States
abramovs@potsdam.edu

Abstract: The paper presents three ways of fair division of circular pizzas among a number of people as a context to teach proper fractions to elementary teacher candidates. One way is to divide each pizza fairly among individuals involved. Another way is informed by an Egyptian fraction representation of a proper fraction developed through the Greedy algorithm. The third way stems from a practical experience developed through one's work in a restaurant. The three ways are compared in terms of the number of pieces each method yields, given the number of pizzas and the number of people. Whereas the first way always yields the largest number of pieces, the relationship between the second and the third ways is complex demonstrating true intricacy of ideas associated with fractions. Activities associated with the last two ways are computationally supported by a spreadsheet and *Wolfram Alpha*.

Key words: common fractions, unit fractions, Egyptian fractions, spreadsheet, *Wolfram Alpha*

1 Introduction

Fractions have been consistently mentioned worldwide as a difficult topic to teach and to learn [14, 4, 13, 7, 11, 10, 9]. Consequently, fraction research (references to a plentiful 20th century research on teaching and learning fractions can be found in the above-cited publications) focused on different aspects of students' and their teachers' conceptual understanding of and procedural competence with this topic of elementary and middle school mathematics. For example, the focus of the most recent publication cited above [9] is the study of tests involving part-whole and measurement interpretations of fractions administered in Malaysia to 10-year old pupils of different achievement levels. Ndalichako [11], analyzing about one million mathematics exams taken at the completion of seven years of primary education in Tanzania, found that examinees lacked conceptual understanding of fractions and incorrectly transferred their knowledge of whole numbers when dealing with mixed fractions. Schifter [14] begins her article by referencing O'Reilly [12], a teacher, who "was stunned and at loss" (cited in [14, p. 55]) after a 6th grader in the United States offered an action-supported interpretation of the fraction 1/2 with which the teacher was not familiar, yet most of the classmates accepted as making sense. Lazić with co-authors [10], reflecting on Serbian experience of teaching part-whole and dividend-divisor models for fractions, described using circles and real-life contexts to demonstrate conceptually the equivalence of unit fractions to a variety of common fractions. Pantziara and Philippou [13], although mentioning that "the fraction 3/4 can be conceived ... as a quotient (three divided by four)", focused their study in Cyprus on other interpretations. Charalambous and Pitta-Pantazi [4], another team of scholars in Cyprus, while considering the dividend-divisor interpretation in the context of fair sharing of pizzas, nonetheless emphasize that "if students are told that three pizzas are evenly distributed among four people, they should be able to identify that the pizzas are shared into fourths and that each person gets three of these shares" (p. 299). This distribution is not the only way to share fairly three pizzas among four people. The present article aims to demonstrate other distributions of pizzas stemming from the dividend-divisor (quotient) interpretation of fractions. It reflects the author's work with elementary teacher candidates, both undergraduates and graduates.

It should also be noted that despite all the research carried out over the years and the latest standards-based recommendations to consider multiple interpretations of fractions including the dividend-divisor interpretation [2, 5, 6], the modern-day elementary teacher candidates are not familiar with this interpretation. At the same time, the dividend-divisor interpretation comes naturally through probing into the properties of division as an operation. In the context of integer arithmetic, when teaching division, a question whether division, like multiplication, is a commutative operation arises. For example, using partition model for division in context, one can conclude that 12 walnuts can be divided among four people fairly. This conclusion can be decontextualized in the form of the equality $12 \div 4 = 3$, representing a simple division fact. However, four walnuts cannot be divided among 12 people. In other words, division is not a commutative operation. Nonetheless, the question whether division is commutative can open window into fractions because, unlike walnuts, not only 12 pizzas can be divided among four people, but, even more realistically, four pizzas can be divided among 12 people as well. The latter gives contextual meaning to the operation $4 \div 12$ without providing a numeric outcome of this operation. The diagram of Figure 1 shows how four pizzas can be divided among 12 people fairly – each pizza is divided into 12 equal slices (pieces).

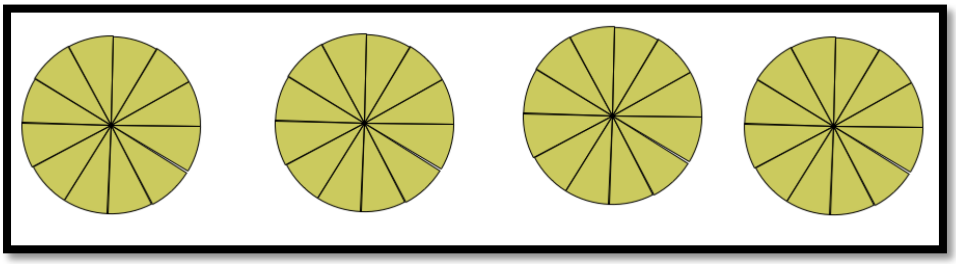

Figure 1. A fair division of four pizzas among 12 people.

Now, using partition model for division in context, the total of 48 pieces, when divided among 12 people fairly, allows each person to get four pizza pieces, something that can be represented numerically in the form $48 \div 12 = 4$. What is the meaning of the number 4 here? These are 4 identical pieces of pizza. While we do not have a numeric name for a piece yet, we know that it is one of 12 identical pieces into which a pizza is divided[1]. Such piece can be given a numerical description in the form of the symbol 1/12, a reciprocal of an integer, commonly called a unit fraction. That is, each person would get four such pieces, something that can be decontextualized to the form of the equality $\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}=\frac{4}{12}$, the right-hand side of

[1] In [10], the circle model was used to introduce unit fractions by dividing circles in 2, 3, 4, 5, and so on equal parts.

which is a notation used to represent the result of repeated addition of four unit fractions as well as the result of dividing 4 by 12. Whereas repeated addition of the reciprocals of integers can be conceptualized as division, the notation for this operation is called a fraction. As suggested in [5], elementary teachers "should focus on developing understanding of fractions, especially unit fractions" (p. 21). The next section will demonstrate a contextual importance of unit fractions.

2 Through transition to unit fractions more effective sharing emerges

Teacher candidates can be asked whether it is possible to try to use measurement model for division when dividing 48 pieces among 12 people. In response, one can suggest putting the pieces in the groups of four; alternatively, measuring pizzas by 4/12 of a pizza, something that is equal, as Figure 2 shows, to 1/3 of a pizza. That is, a serving for each person would be 1/3 of a pizza and having 3 servings from each of the 4 pizzas gives 12 servings (Figure 2). Decontextualization implies the relation of equivalence, 4/12 = 1/3, between two fractions which makes sense if we interpret its right- and left-hand sides, respectively, as sharing one pizza with three people and sharing four pizzas with 12 people. Indeed, sharing each of the four pizzas with three people means sharing the pizzas with 12 people. This interpaly between concrete and abstract, between "the ability to *contextualize* ... in order to probe into the referents for the symbols involved ... [and] the ability to *decondtextualize* – to abstract a given situation and represent it symbolically" [5, p. 6, italics in the original], or, in the words of Vygotsky [16], between "the first-order symbols ... [and] the second-order symbolism" (p. 115), is supposed to show a friendly face and factual meaning of the reduction of fractions to the simplest form.

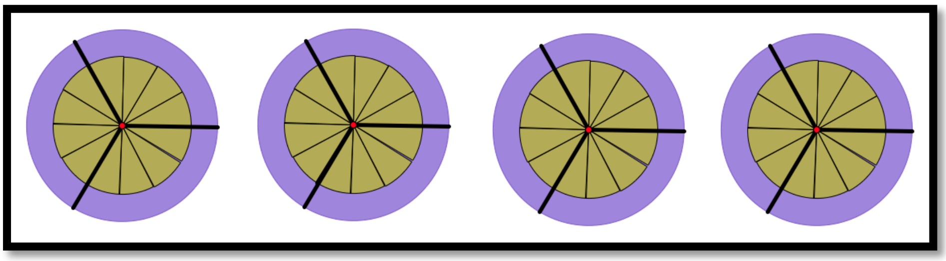

Figure 2. Measuring pizzas by 1/3 of pizza.

As shown in Figure 1, dividing 4 pizzas into 12 pieces each yields 48 pieces. At the same time, dividing 4 pizzas into 3 pieces each yields 12 pieces – significantly smaller number of pieces than in the former case. This suggests that when a non-unit fraction can be replaced by a unit fraction, due to the equivalence of the two fractions, this replacement means, contextually, that unit fractions can somehow inform a strategy of sharing pizzas among people with the smaller number of pieces than in the former case. With this in mind, consider the case of dividing 5 pizzas among 6 people. A straightforward way is to divide each pizza into 6 pieces as shown in Figure 3 to get 30 pieces, so that each person would get 5/6 of a pizza. Could the sharing be done more effectively in s sense of having fewer than 30 pieces? Figure 4 shows such division – each of the three pizzas is divided in half and the remaining two pizzas are divided in three equal parts with the total number of pieces equal to 12. That is, each person would get 1/2 of a pizza and 1/3 of a pizza. Decontextualization from pizzas, used as the first-order symbols, results in the relation $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$, representing the second-order symbolism. Once again, we see that the appearance of unit fractions is responsible for the smaller number of pieces in comparison with the straightforward division. This time, although a non-unit fraction 5/6 is already in the simplest form, it can be replaced by a sum of two *different* unit fractions which, alternatively, informs the context of dividing 5 pizzas among 6 people. When such way of dividing pizzas is found, one comes across a very interesting strand of the history of mathematics – an Egyptian fraction. Known for almost four millennia, this name is reserved for a finite sum of distinct reciprocals of positive integers. A table, representing fractions of the form 2/*n* as a sum of distinct unit fractions appeared in the famous Egyptian papyrus roll (ca. 1650 B.C.) found in 1858 by Henry Rhind, a Scottish scholar and collector of antiques [3]. The table is available online at

https://en.wikipedia.org/wiki/Rhind_Mathematical_Papyrus_2/n_table. Although the genesis of using Egyptian fractions is not known, several insightful suggestions of how the representations for that table were found are offered at the above-mentioned website. From a mathematics education perspective, a history of ancient Egyptian mathematics provides context for learning fractions through dividing pizzas – a familiar activity to every student. This paper is written to share didactical ideas about teaching fractions in the context of pizza sharing informed by Egyptian fractions. So, André Weil's notable remark that ancient Egyptians "*took a wrong turn*" [8, p. 290, italics in the original] when using their notation to represent proper fractions may be revisited from a mathematics education perspective.

3 Egyptian fractions and the Greedy algorithm

An Egyptian fraction is a sum of a finite number of distinct unit fractions. For example, $\frac{1}{2}+\frac{1}{3}$ and $\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$ are Egyptian fractions. A unit fraction itself, like $\frac{1}{3}$, is an Egyptian fraction, although it can be represented as a sum of other unit fractions in many different ways; for example, $\frac{1}{3}=\frac{1}{4}+\frac{1}{13}+\frac{1}{156}$. Sometimes, an Egyptian fraction is understood as a *special* representation of a fraction (alternatively, a positive rational number) in the form of a finite sum of distinct unit fractions derived through a special algorithm, so that the sum $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$ which is equal to $\frac{71}{105}$ may not be considered an Egyptian fraction representation of $\frac{71}{105}$. Indeed, if one asks *Wolfram Alpha* to represent $\frac{71}{105}$ as an Egyptian fraction, the answer is $\frac{1}{2}+\frac{1}{6}+\frac{1}{105}$. However, the term Egyptian fraction can be used for any finite sum of distinct unit fractions.

Apparently, *Wolfram Alpha* utilizes the so-called Greedy algorithm for developing Egyptian fractions. This algorithm was first used by Fibonacci to convert a given proper non-unit fraction into an Egyptian fraction. The algorithm is based on finding the largest unit fraction smaller than the given fraction, then finding the difference between the two fractions and continuing (if the difference is not a unit fraction) the algorithm with the difference. For example, the largest unit fraction smaller than $\frac{3}{5}$ is $\frac{1}{2}$. Thus $\frac{3}{5}=\frac{1}{2}+x$, where $x=\frac{3}{5}-\frac{1}{2}=\frac{6}{10}-\frac{5}{10}=\frac{1}{10}$. So, $\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$, where $\frac{1}{2}+\frac{1}{10}$ is an Egyptian fraction representation of $\frac{3}{5}$. A non-Greedy algorithm could be to write $\frac{3}{5}=\frac{1}{3}+x$, where $\frac{1}{3}$ is not the largest unit fraction smaller than $\frac{3}{5}$. We have $x=\frac{3}{5}-\frac{1}{3}=\frac{9}{15}-\frac{5}{15}=\frac{4}{15}$. That is, $\frac{3}{5}=\frac{1}{3}+\frac{4}{15}$. In turn, $\frac{4}{15}=\frac{1}{4}+x$ (here we use the Greedy algorithm as $\frac{1}{4}$ is the largest unit fraction smaller than $\frac{4}{15}$), where $x=\frac{4}{15}-\frac{1}{4}=\frac{16}{60}-\frac{15}{60}=\frac{1}{60}$. Alternatively, $\frac{4}{15}=\frac{1}{5}+x$, where $x=\frac{4}{15}-\frac{1}{5}=\frac{4}{15}-\frac{3}{15}=\frac{1}{15}$. That is, $\frac{1}{3}+\frac{1}{4}+\frac{1}{60}$ and $\frac{1}{3}+\frac{1}{5}+\frac{1}{15}$ are another two Egyptian fraction representation of the fraction $\frac{3}{5}$. One can also see that Egyptian fraction representation of a common fraction is not unique.

4 Three ways of dividing pizzas

It should be noted that whereas both the straightforward division and Egyptian fraction division of pizzas provide each person with *identical* pieces (either uniquely identical like in the former case or identical in terms of a number of pieces like in the latter case), it is possible to divide pizzas fairly in terms of the quantity of pizza, yet without providing each person with the same number of pieces. Such division may be called a *semi-fair* division. Figure 5 shows a semi-fair division of 5 pizzas among 6 people by cutting off an identical piece (measuring 1/6 of a pizza) from each pizza enabling one person to get 5 such pieces and each of the remaining 5 people to get the rest of each pizza (measuring 5/6 of a pizza). One can note that the total number of pieces in which 5 pizzas were divided through this method is 10 – smaller than in the case of the straightforward division (30 pieces) and in the case of Egyptian fraction division (12 pieces). It is interesting to note that the author learned about this method of dividing pizzas from a manager of an eating establishment when the former asked the latter how to divide pizzas in a smaller number of pieces than an obvious straightforward division provides. The intent of the author was to see the birth of Egyptian fractions in the practice of real life. Instead, the author encountered what may be called practice-fostered mathematics. This account has important implications for mathematics education for it signifies that mathematical ideas often emerge as a reflection on common sense and real-life experience associated, in the words of Vygotsky [16], with the first-order symbols and then become abstracted to the level of the second-order symbolism. One can only guess if some 36 centuries ago the use of Egyptian fractions was motivated by something purely contextual and then recorded by Ahmes (an ancient scribe, the first known contributor to mathematics) on the Rhind papyrus roll. At the same time, it appears that representations by Ahmes of the fractions of the form $2/n$ through a sum of distinct unit fractions was not motivated by the need of effective sharing. For example, the Greedy algorithm yields $\frac{2}{13}=\frac{1}{7}+\frac{1}{91}$ and the papyrus roll has $\frac{2}{13}=\frac{1}{8}+\frac{1}{52}+\frac{1}{104}$. Interpreting each representation as dividing 2 pizzas among 13 people, indicates 26 and 39 pieces, respectively. One can check to see that the semi-fair division gives 24 pieces (see the spreadsheet of Figure 10, cell X2).

It should be noted that the Greedy algorithm may produce more unit fractions than a non-Greedy algorithm in converting a common fraction into an Egyptian fraction. For example, according to *Wolfram Alpha*, $\frac{5}{21}=\frac{1}{5}+\frac{1}{27}+\frac{1}{945}$, where $\frac{1}{5}$ is the largest unit fraction smaller than $\frac{5}{21}$. At the same time, $\frac{5}{21}=\frac{1}{6}+\frac{1}{14}$ and $\frac{1}{6}<\frac{1}{5}$. That is, the latter conversion of $\frac{5}{21}$ into an Egyptian fraction was developed through a non-Greedy algorithm and it consists of fewer unit fractions than the former conversion. Consequently, when sharing 5 pizzas among 21 people, the number of pizza pieces is larger in the case of the former division (63) than in the case of the latter division (42).

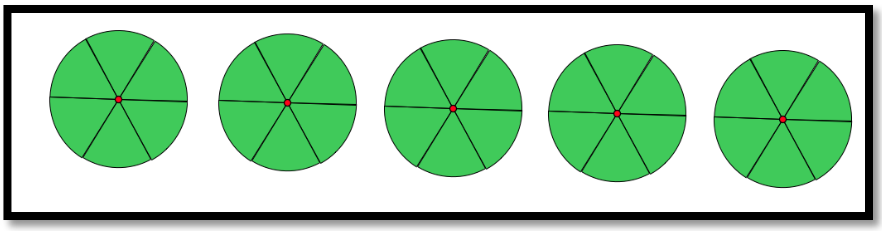

Figure 3. There are 30 pizza pieces to share with 6 people.

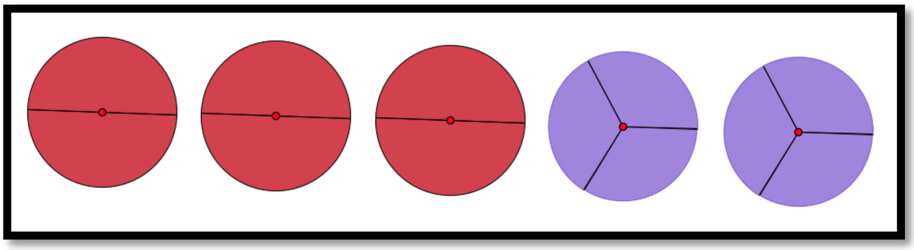

Figure 4. There are 12 pizza pieces to share with 6 people.

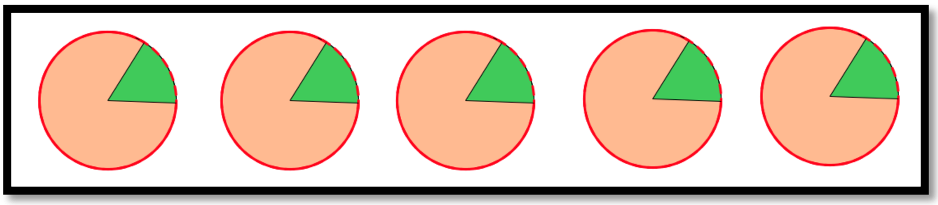

Figure 5. There are 10 pizza pieces to share with 6 people.

5 Two more examples

Two more examples have to be considered in order to demonstrate a non-linear relationship between the methods of division – the Greedy algorithm Egyptian and the semi-fair – and the fractions representing pizzas and people. Consider the case of dividing 4 pizzas among 7 people using the two methods. The relationship $\frac{4}{7}=\frac{1}{2}+\frac{1}{14}$, obtained through the Greedy algorithm as $\frac{4}{7}>\frac{4}{8}=\frac{1}{2}$ and $\frac{4}{7}-\frac{1}{2}=\frac{1}{14}$, means that each person out of 7 would get 2 pieces – half a pizza and 1/14 of a pizza. This division yields 14 pieces, something that can also be seen in Figure 6. At the same time, the semi-fair method shown in Figure 7 yields the total of 16 pieces: 12 pieces each measuring 1/7 of a pizza - these can ne divided among 3 people so that each gets 1/7 + 1/7 + 1/7, and 4 pieces each measuring 4/7 of a pizza. That is, when dividing 4 pizzas among 7 people, we have more pieces when applying the semi-fair method than the Egyptian method. At the same time, dividing 5 pizzas among 7 people by using the two methods yields a different relationship between the number of pieces obtained in each case. *Wolfram Alpha* yields $\frac{5}{7}=\frac{1}{2}+\frac{1}{5}+\frac{1}{70}$ meaning that each person gets three different pieces so that 7 people get 21 pieces. The semi-fair division, as shown in Figure 8, yields 15 pieces; that is, we have fewer pieces when applying the semi-fair method than the Egyptian fraction method.

Whereas the relationship between two methods is difficult to describe, the semi-fair division seems to be following a pattern. Indeed, when sharing 4 pizzas among 7 people, 3 people get 4/7 of a pizza through iterating [15] four times a piece measuring 1/7 of a pizza and 4 people get 4/7 of a pizza through a single piece. When sharing 5 pizzas among 7 people, 2 people get 5/7 of a pizza through 5 pieces measuring 1/7 of a pizza (i.e., iterating five times a 1/7 piece) and 5 people get 5/7 of a pizza through a single piece. In the first case, we have 3 + 4 = 7 and $3\cdot\left(\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}\right)+4\cdot\frac{4}{7}=\frac{4}{7}\cdot(3+4)=4$. In the second case, we have

$$2+5=7 \text{ and } 2\cdot\left(\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}\right)+5\cdot\frac{5}{7}=\frac{5}{7}\cdot(2+5)=7.$$

Similarly, when dividing 5 pizzas among 6 people we had 1 + 5 = 6 and

$$1\cdot\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)+5\cdot\frac{5}{6}=\frac{5}{6}\cdot(1+5)=5.$$

The three special cases provide an empirical evidence for generalization.

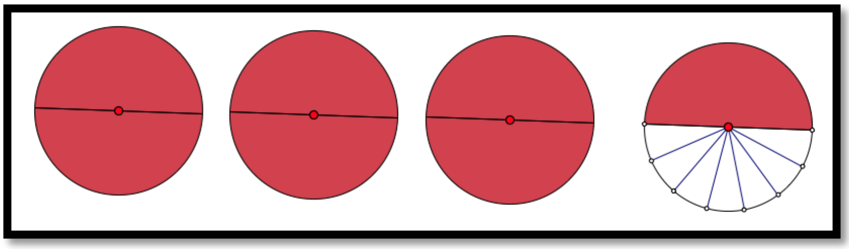

Figure 6. Egyptian fraction division of 4 pizzas among 7 people

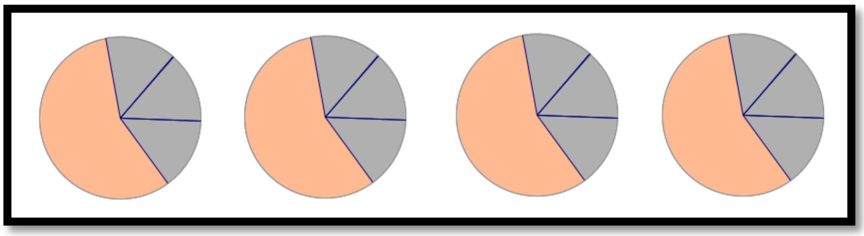

Figure 7. Semi-fair division of 4 pizzas among 7 people

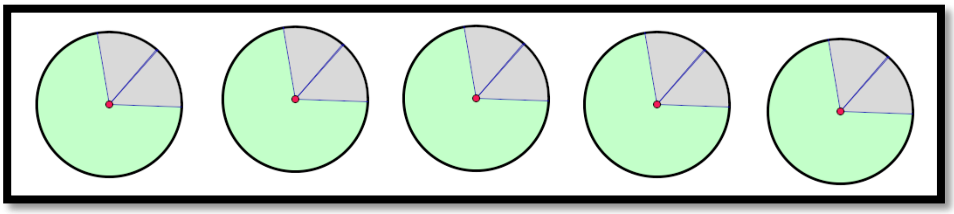

Figure 8. Semi-fair division of 5 pizzas among 7 people.

6 The joint use of *Wolfram Alpha* and a spreadsheet

In the case of dividing m pizzas among n people, $m < n$, through the semi-fair method we have each pizza divided into $n - m + 1$ pieces, $n - m$ of which measured $1/n$ of pizza and one piece measured m/n of pizza. Therefore, the total number of pieces is equal to $m(n - m + 1)$ and when put together make up m pizzas. Indeed, $m\left[(n-m)\cdot\frac{1}{n}+\frac{m}{n}\right]=m\cdot n\cdot\frac{1}{n}=m$. For example, when $m = 4$ and $n = 7$ we have 16 pieces in all: for each of the 4 pizzas there are 3 pieces measuring 1/7 of a pizza, that is, the total of 12 such pieces – enough to share fairly among 3 people, and 1 piece measuring 4/7 of a pizza, that is, the total of 4 pieces – enough to share with the remaining 4 people. Furthermore, the difference between the number of pieces obtained through the straightforward division, nm, and the number of pieces obtained through the semi-fair division, $m(n - m + 1)$, does not depend on the number of people, n. Indeed, $nm-m(n-m+1)=m^2-m$. At the same time, such difference in the case of the straightforward and (the Greedy) Egyptian divisions depends on the number of people. For example, because $\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$ and $\frac{3}{7}=\frac{1}{3}+\frac{1}{11}+\frac{1}{231}$, in the case of dividing 3 pizzas among 5 people the difference is 15 – 10 = 5 and in the case of dividing 3 pizzas among 7 people the difference is 21 – 21 = 0.

One can integrate *Wolfram Alpha* and a spreadsheet to study numerically the relationship between the number of pieces into which m pizzas can be divided among n people, $n > m$, assuming that GCD(n, m) = 1. Such a spreadsheet is called an integrated spreadsheet [1]. By analyzing numerical data, one can make the following observation. The numbers in the columns for semi-fair division display a very simple pattern: numbers equidistant from the top and the bottom of a column are the same. For example, when 2 pizzas are divided among 7 people and when 6 pizzas are divided among 7 people, the resulting number of pieces is the same in both cases (Figure 9). Here 7 – 6 + 1 = 2. Likewise, when 2 pizzas are divided among 11 people and when 10 pizzas are divided among 11 people, the resulting number of pieces is the same in both cases (Figure 10). Here 11 – 10 + 1 = 2. Also, when 3 pizzas are divided among 13 people and when 11 pizzas are divided among 13 people, the resulting number of pieces is the same in both cases (Figure 10). Here 13 – 11 + 1 = 3. These special cases can be generalized algebraically. Indeed, when m pizzas are divided among n people and $n - m + 1$ pizzas are divided among n people in the semi-fair way, the division yields the same number of pizza pieces: in the first case the number is equal to $m(n - m + 1)$ and in the second case – $(n - m + 1)[n - (n - m + 1) + 1] = (n - m + 1)m$.

	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1		3	E	4	E	5	E	6	E	7	E	8	E	9	E	10	E
2		4	6			8	10			12	14			16	18		
3				6	8	9	10			15	21	18	16			24	20
4						8	15			16	14			24	18		
5								10	12	15	21	20	16	25	18		
6										12	21						
7												14	24	21	27	28	20
8														16	27		
9																18	30

Figure 9. An integrated spreadsheet 1.

	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE	AF	AG	AH	AI	AJ	AK	AL	AM
1	11	E	12	E	13	E	14	E	15	E	16	E	17	E	18	E	19		20	E
2	20	22			24	26			28	30			32	34			36	38		
3	27	22			33	39	36	28			42	32	45	34			51	57	54	40
4	32	22			40	39			48	30			56	68			64	38		
5	35	33	40	24	45	39	50	28			60	32	65	51	70	36	75	38		
6	36	22			48	39							72	34			84	57		
7	35	33	42	24	49	26			63	45	70	48	77	51	84	36	91	57	98	40
8	32	44			48	39			64	30			80	68			96	57		
9	27	44			45	39	54	28			72	32	81	34			99	76	108	60
10	20	44			40	39							80	51			100	38		
11			22	36	33	39	44	42	55	45	66	48	77	51	88	36	99	57	110	40
12					24	52							72	51			96	57		
13							26	56	39	45	52	48	65	51	78	54	91	57	104	60
14									28	45			56	68			84	95		
15											30	64	45	68			75	76		
16													32	85			64	57		
17															34	54	51	95	68	60
18																	36	76		
19																			38	80

Figure 10. An integrated spreadsheet 2.

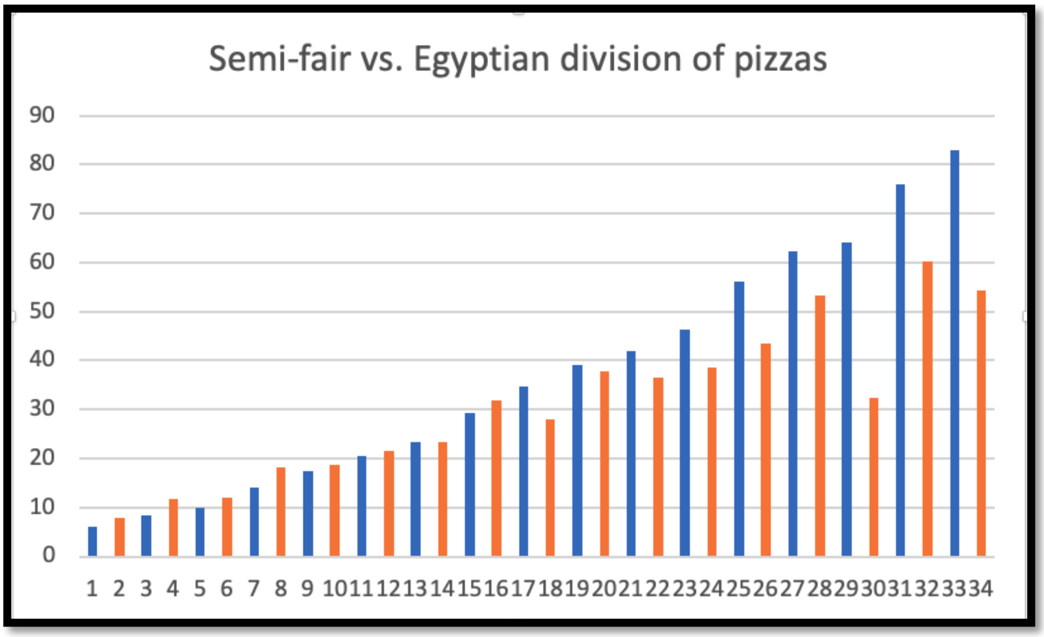

Figure 11. Odd and even bars represent, respectively, semi-fair and Egyptian divisions.

7 Conclusion

This paper was written to share the author's experience of using the context of cutting pizzas into equal quantities to teach fractions to elementary teacher candidates. The traditional way of teaching fractions is through cutting a single pizza into equal pieces using the part-whole interpretation of a fraction to demonstrate the meaning of a proper fraction, including a unit fraction. A proper fraction can also be given meaning through the use of more than one pizza; for example, the fraction $\frac{m}{n}, m < n$, can be introduced using the dividend-divisor (quotient) model by dividing m pizzas among n people. In this case, pizzas are typically cut into equal pieces.

The quotient model can be extended with a dual goal: to allow for having equal *quantities* made of pizza pieces, not just equal pieces, and to introduce two educationally important contexts as alternatives to the part-whole and quotient models. In this paper, one such context was history-oriented going back several millennia of mathematical development, and another context was practice-oriented with genesis in the modern day fast food routine. The former context included Egyptian fractions the introduction of which may be motivated by trying to minimize the total number of pizza pieces resulting from fair sharing of pizzas among people. The latter context, in some cases, was shown providing unexpectedly a more efficient way of cutting pizzas into pieces by minimizing their number in comparison with the former context. This efficiency, however, as was shown in the paper, depends on the values of m (pizzas) and n (people). Pedagogically, both contexts are important as teacher candidates and their future students alike need to know history of the subject matter and to appreciate that the origin of mathematics is in a practical activity.

The paper demonstrated the joint use of a spreadsheet and *Wolfram Alpha*. The use of the spreadsheet made it possible to support numeric modeling of what was called the practice-fostered mathematics (semi-fair division of pizzas among people) when practical experience was abstracted to offer a formal mathematical rule enabling uncomplicated programming. The use of *Wolfram Alpha* allowed one to generate Egyptian fraction representations of common fractions through the Greedy algorithm, something that was not available within a spreadsheet. An interesting aspect of the joint use of two computational tools was the absence of any pattern in the relationship between the two contexts. Whereas the practice-fostered mathematics enabled algebraic generalization that was used in spreadsheet modeling, a more mathematically oriented context of Egyptian fractions did not allow for any generalization and demonstrated real complexity of mathematics associated with fractions. Apparently, this complexity is not a surprise for a professional mathematician. However, it is likely to be an eye-opener for a prospective elementary teacher who believes that the only challenging aspect of fractions deals with the need to do four arithmetical operations. Teaching fractions through cutting pizzas into equal quantities makes it possible to derive rather intricate additive decompositions of common fractions through numeric description of physically created images under the umbrella of common sense. This real-life appeal of elementary mathematics can be brought to light through teaching the entire curriculum of the subject matter for it motivates one's natural curiosity and uplifts hidden creativity of teacher candidates and their future students alike.

Acknowledgement. The author wishes to acknowledge the authorship by Dmitry Sozaev of the method of dividing pizzas called in this paper the practice-fostered mathematics.

References

[1] Abramovich, S. (2016). *Exploring Mathematics with Integrated Spreadsheets in Teacher Education*. Singapore: World Scientific.

[2] Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics. https://amte.net/standards.* Accessed 25 October 2020.

[3] Chace, A.B., Manning, H.P., Archibald, R. C. (1927). *The Rhind Mathematical Papyrus*, *British Museum 10057 and 10058*, vol. *1*. Oberlin, OH: Mathematical Association of America.

[4] Charalambous, C. Y., and Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fraction. *Educational Studies in Mathematics*, **64**(3): 293–316.

[5] Common Core State Standards. (2010). *Common Core Standards Initiative: Preparing America's Students for College and Career.* http://www.corestandards.org.

[6] Conference Board of the Mathematical Sciences. (2012). *The Mathematical Education of Teachers II*. Washington, DC: The Mathematical Association of America.

[7] Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B. and Content, A. (2013) A componential view of children's difficulties in learning fractions. *Frontiers in Psychol*ogy, **4** (Art. 715): 1-12.

[8] Graham, R. (2013). Paul Erdős and Egyptian fractions. In L. Lovász, I.Z. Ruzsa, V.T. Sós, D. Palvolgyi (Eds.), *Erdős Centennial* (pp. 289–309). Heidelberg, Germany: Springer-Verlag and János Bolyai Mathematical Society.

[9] Kor, L.-K., Teoh, S.-H., Mohamed, S.S.E.B., and Singh, P. (2019). Learning to make sense of fractions: Some insights from the Malaysian primary 4 pupils. *International Electronic Journal of Mathematics Education*, **14**(1): 169-182.

[10] Lazić, B., Abramovich, S., Mrda, M, and Romano, D. A. (2017). On the teaching and learning of fractions through a conceptual generalization approach. *IEJME–Mathematics Education*, **12**(8): 749-767.

[11] Ndalichako, J. L. (2013). Analysis of pupils' difficulties in solving questions related to fractions: The case of primary school leaving examination in Tanzania. *Creative Education*, **4**(9): 69-73.

[12] O'Reilly, A.M. (1996). Understanding teaching/teaching for understanding. In D. Schifter (Ed.), *What's Happening in Math Class?*, Volume 2: Reconstructing professional identities (pp. 65–73). New York: Teachers College Press

[13] Pantziara, M., and Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, **79**(1): 61–83.

[14] Schifter, D. (1998). Learning mathematics for teaching: From a teacher's seminar to the classroom. *Journal of Mathematics Teacher Education*, **1**(1): 55–87.

[15] Steffe, L., and Olive, J. (2010). *Children's Fractional Knowledge*. New York, NY: Springer.

[16] Vygotsky, L.S. (1978). *Mind in Society* Cambridge, MA: Harvard University Press.