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The Orbital Torus of Arrokoth -- A Sample Project

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Abstract. In mathematics education there has been voluminous consideration of "alternative" approaches to deliver content. At every stage of formal education, motivating students to learn mathematics is accomplished by making the material relevant to them. In the undergraduate calculus sequence, concepts of mensuration are made "real" when students actively use the relevant processes. While textbooks generally offer examples and exercises for students, these are more effective when students have personal motivation via Action Learning, or what are often called "Projects."

The article presented here is an example of what might be composed by the student as an application project. It demonstrates that, in addition to the mathematics, there is an abundance of relevant discussion connected to a topic to gain the writer's (and hopefully the reader's, as well) interest. Projects are reviewed by course instructors and subject area supervisors. The *one* student plus *two* professionals requirement has merit supported by this example. For starters, the instructor need not be (in this case) a professional astronomer and the astronomers (subject area advisors) need not know all the mathematical details behind each equation presented in a project write-up. The student can draw from these two real sources for motivation, and there's "action" for all participants. It is not difficult to sense that a project student will likely retain the information learned for a longer period and be ready to synthesize the concepts with other real-world situations as they may arise. This is the desired outcome in mathematics education, for the student has learned the material, if the student can apply the material. The results from Projects are pragmatically sound. In a nutshell, it works.

Keywords: Mathematical projects, Education with application, Symmetry, Volume calculation

Introduction

Undergraduate Calculus courses are known to be quite challenging. By including Projects, more students can be successful. Additionally, students attain higher levels of understanding when they apply mathematical concepts to their personal work. This conclusion is found in various studies such as "Implementing Projects in Calculus on a Large Scale at the University of South Florida" [5] and "Teaching Mathematics through Concept Motivation and Action Learning" [1]. Project work is deservedly called "action learning," for it requires the application of mathematical concepts to real-world situations. Many of the best projects are shared in the *Undergraduate Journal of Mathematical Modeling: One + Two* (UJMM) [18] repository. This journal is a splendid resource for future action learning. Readers will be inspired by articles in UJMM in different ways and to varying degrees. Since there are generally further physical and mathematical implications to any study, readers can freely draw up the UJMM presentations (with proper attributions to the authors) and perhaps develop their own action learning.

While the UJMM audience is likely comprised largely of undergraduate students, others may consider these works as well. The present article emphasizes that project ideas can be extended. This writing makes use of *Calculating the Area within the Orbit of Arrokoth* by A. Collins [3], and demonstrates the "extended action learning" possible from that project. In particular, the notion of area is extended to volume. Some simplified general observations about solar system bodies are made. A discussion of "planet" status is also included in what follows.

Materials and Methods

At the University of South Florida, the administration sees fit to allow instructors of certain undergraduate mathematics courses to decide whether to employ Projects. Also, qualifying instructors electing to use Projects can determine the weight given to this portion relative to all required course testing. One strategy is to allow students to have project work replace the course final examination. This provides an alternative to "pure testing" from which some students can benefit. Projects also provide a fuller, working understanding of the concepts involved.

The concept of symmetry is one which resurfaces constantly throughout the K-16 educational spectrum. Calculus offers very concise formulas for the determination of areas and volumes of ellipses and tori, and these formulas often involve symmetry. There may be some subset of what follows which would serve as a sample Project for elementary or secondary levels of mathematics education. The following is directed toward students in the latter part of their undergraduate Calculus sequences. Elementary students, as well as those at secondary and tertiary levels of formal mathematics education can gain from consideration of symmetry associated with certain "ovoid" or elliptic shapes. The "student" may certainly have completed "16+" of the K-16 formal years and still find some extended action learning value, say, in symmetry.

The area within the ellipse of the Kuiper Belt object $2014MU_{69}$ was first calculated (in [3]). On January 1, 2019, the New Horizons spacecraft flew past the Kuiper Belt object which has since been named Arrokoth. The position of $2014MU_{69}$ is indicated in Figures 1 and 2 below. The area enclosed by the orbit may be of interest. Going on that assumption, the orbital tori of Arrokoth, Earth, and Hale-Bopp are considered. There are definite problems to measurement which are noted, but largely ignored. The rationale for a simplified model is that it should still give some sense of the full physical condition. Even the "best" models will always be deficient in some way. Part of the motivation is to make this point concerning the unavoidable deficiencies in any model. Especially in astronomical situations there are details which any model will have to defer to the prototype. Still, we know that NASA's model was sufficient to deliver the New Horizon's probe to the vicinity of Pluto and then Arrokoth. Evidently, being able to adjust the trajectory of the spacecraft during flight was necessary to get this done.

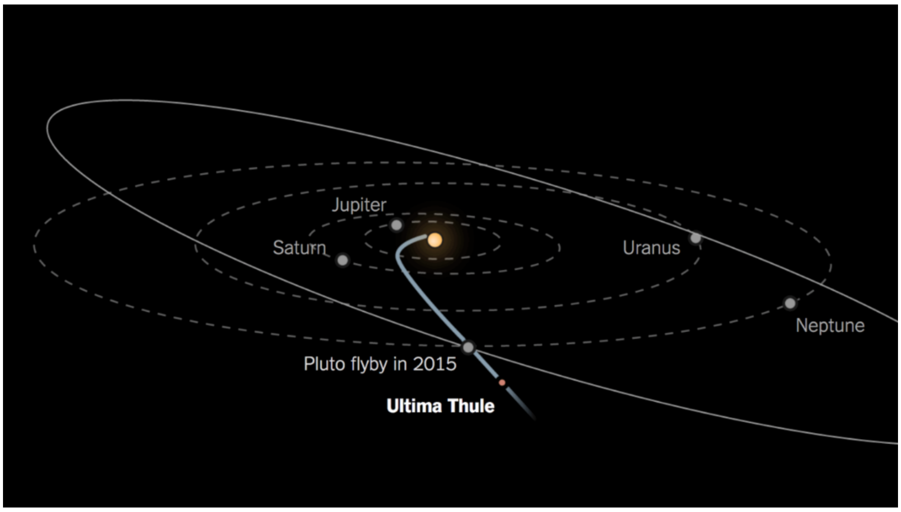

Figure 1. Map for the location of *2014MU*$_{69}$.

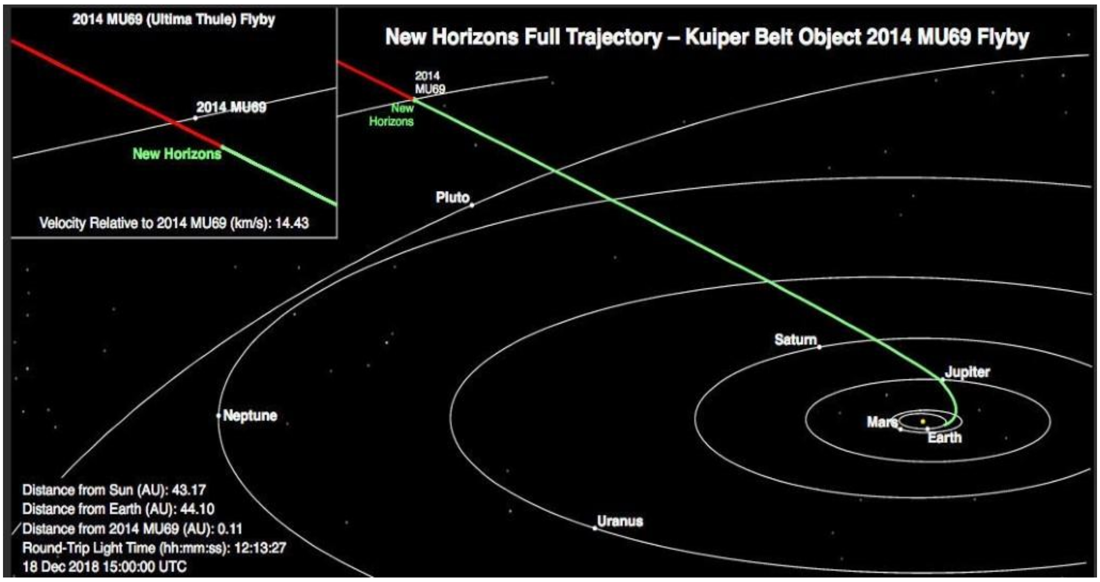

Figure 2. Contact of New Horizons with Arrokoth at the beginning of 2019.

Figures 1 and 2 above remind us that the New Horizons journey began in January of 2006, flew by Pluto in 2015, and then by Arrokoth on January 1, 2019. It's still going strong in 2020. It appears to have been a good decision not to terminate the mission at Pluto, nor at Arrokoth. Instead, New Horizons is the proverbial bear going over the mountain. It gets to see what more it can see.

As observed in [3], there is a lot of interest for this object since it is new to the astronomical community and is one of the furthest objects to be studied by a probe in our solar system. Therefore, this otherwise indistinct object, once called 2014MU$_{69}$, is worthy of consideration. The object also orbits the Sun, so certain calculations for Arrokoth can be compared to those of other solar system objects. Knowing the area under the ellipse of an object may be important to the astronomical community, including research into particle physics with an emphasis on dark matter and neutrino masses [3]. Finding the area within an ellipse would be the first step to understanding what can be held within that area. Further investigation into the Arrokoth object can be easily facilitated by considering available reports [2, 8, 9, 12].

It is of interest that in the thirteen years it took to reach Arrokoth, processors have advanced to the point that 3D animations are now more easily rendered on computer (or phone) screens. There is no reason to restrict models to planar conditions when there are truly three dimensions involved. It's often convenient to view a simplified model in a plane, since many planetary bodies tend to follow the Sun's equator. Since there are new things of interest when we look at objects from above and below their poles, we should not exclusively study equatorial regions. The unexpected ice on the moon and Mars, the curious distortion evident in Jupiter's magnetic field, and the odd geometry of Saturn's pole, are examples. Certainly, polar regions should not be ignored. It is of interest that Earth's polar regions demonstrate extremes of climate as do the polar regions of other planets. Interestingly, no region on Earth appears to be too extreme for life.

As noted, there is always something to be learned about symmetry. Symmetry turns out to be a very important concept in Calculus. One might also ponder whether the symmetries of the ellipse extend in any meaningful way to those exhibited by a plot describing analytic continuation. Figure 3 is perhaps a relevant example to consider. Motion above and below the plane may have components which are best modeled using other ideas of mathematics, even perhaps "analytic continuation" [3].

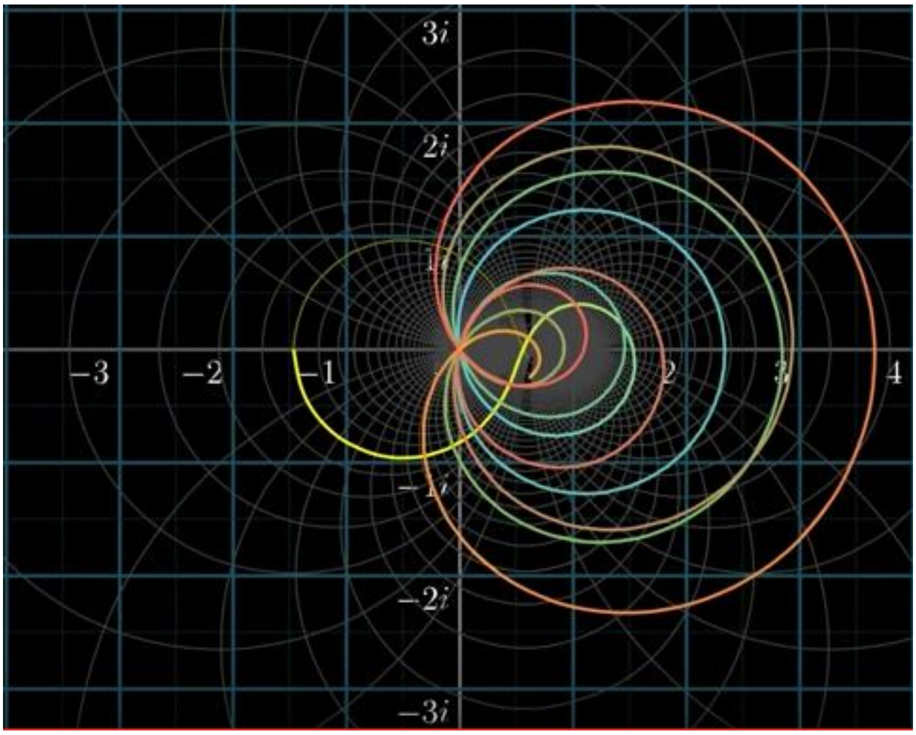

Figure 3. Diagram of analytical continuation.

Results

Generally, there are uncertainties, as well as limitations, to what can be observed and measured. There may also be questions as to *what* should be measured. Many of the same philosophical questions pondered by the Ancients and again by the enlightened thinkers of more recent times are still with us. We owe it to ourselves to keep asking those questions and hoping to better understand the human condition while pondering the Universe. Since the development of Newtonian mechanics and Leibniz notation, there has been much success in applying mathematics to solar-bound as well as to earthbound geometry. Notice that the very word "geometry" is somewhat archaic, since the word suggests "Earth measuring". We might tolerate the use of "astrometry" (Merriam-Webster says that its first known use was in 1811), just as we accept "astroid" to describe the shape of a diamond with parabolic edges. They are interesting, culturally grown words.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

is the mathematical model of an ellipse. Equation (1) can be used with Kepler's Laws of planetary orbital motion. Solving for y in equation (1) yields the equation

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2} \tag{2}$$

which is the relation used for integration. Since we have symmetry with respect to both the x- and y-axes, we need only consider one quarter (pragmatically, the first quadrant) when taking the integral as follows.

$$\frac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx \tag{3}$$

Stewart [15] shows how to substitute $x = a\sin\theta$ in equation (3), and use $\sqrt{a^2 - x^2} = a\cos\theta$ and $dx = a\cos\theta\, d\theta$ as follows:

$$\begin{aligned} A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx \\ &= 4\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\, d\theta \\ &= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta \\ &= 4ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta)\ d\theta \end{aligned}$$

$$A = \pi ab \approx 6249.286 \text{ AU}^2 \tag{4}$$

Result (4) allows us to say that the area within Arrokoth's orbit is about 2000 times that of Earth's ellipse (at about 3.14116 AU^2). Astronomical Units (AU) are used for convenience. An Astronomical Unit is the mean Earth – Sun distance. Since the orbit is not well known, there might be a 2 or 3% error in the measured focal length, semi-major axis, and semi-minor axis used for Arrokoth. Incidentally, it seems that Earth and Arrokoth have roughly the same eccentricity. (Eccentricity being the deviation of an orbit from circularity.) The eccentricity details might be a subject for future research.

There are many questions to ponder. For example, we may wonder, as did Kepler, '*Why we have the interplanetary spacings that we do*?' Or ask, '*What significance does orbital inclination have*?' Or, '*Why are the observed focal lengths what they are*?' These queries may suggest possible topics for future study, as well.

It seems that by now considering the volumes relative to some of the area findings, we might get a fuller sense of the vastness of interplanetary space. The Sun's system is evidently just that, namely a *system* where all the bodies orbiting the Sun are intricately connected. The connection is at least gravitational, however there may be forces at work which we know nothing about or are just beginning to glean. It may be reasonable to consider such basic concepts as areas or volumes of planetary orbits as initial findings.

One criterion applied to the distinction of "planet" is the clearing out of debris from the path of the orbiting body. Presumably, a body will clear out a roughly circular toroidal region centered on the Sun. The volumes of the planetary tori would seem to have theoretical interest. It is likely that others have speculated on this and related notions, however this notion seems like it might not be well-studied. It's fair to consider how ideas in [3] might be extended in a "casual" fashion. We can presume that there exists a circular region about a body's mean orbital path representing the extremes of its possible orbital positions. This can be visualized as in Figure 4.

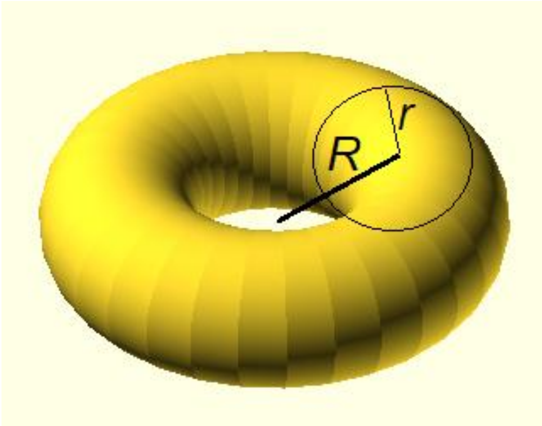

Figure 4. Hypothetical toroidal orbital band with the Sun at the center.

The toroidal band in Figure 4 is a "rotational extrusion" of the circle of extremes with radius r and at an average distance of R from the Sun. The Sun is at the center and the points on the torus nearest the Sun are places where perihelion occurs and those points farthest from the Sun would be about where aphelion would occur. The model assumes that over the extended life of the body, all the possible positions have been attained (sweeping out any interplanetary material). All the celestial motions mentioned in [3] are at work here (and probably others which are not mentioned). Since the assumptions would seem acceptable, we might then ask what the volume of such tori are for planetary bodies such as Earth and Arrokoth.

It is understood that Arrokoth, being only about 15 to 22 km across, is far from being a planet. Arrokoth is a planetoid or dwarf planet, at best. Never-the-less it is like Earth and other planets in being gravitationally bound to the Sun. The objects within the Kuiper belt are evidently not unlike those occupying the asteroid belt, at least in size and in their inability to clear their paths (or tori).

An interesting aside, perhaps, is the continuing redefinition of "planet." It seems we once believed that the Sun was a planet, along with the moon and the five other wanderers which appeared to circle Earth (which was not then itself a planet). The seven planets are immortalized on the seven-day calendar. With the development of telescopes and mathematics we've settled on eight. We should get used to changes in understanding and not let the demotion of Pluto disturb us.

It has been popularly reported that hundreds of extra-solar planets have now been inferred. It may be that there are yet more large, but dark, solar planets to add to our eight at some point. In the early nineteenth century, Ceres and other asteroids were discovered.

These were first taken to be planets. We've had to demote planets before, so it's not so strange to do so again.

Most minor planets, as well as Earth, have small eccentricities. It appears that the described theory results in relatively small toroidal radii when the eccentricity, e, is small. Referring again to Stewart [15], there is this nice unifying theorem on conic sections:

> Let F be a fixed point (called the focus) and l be a fixed line (called the directrix) in a plane. Let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that
>
> $$\frac{|PF|}{|Pl|} = e$$
>
> (that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is
>
> (a) an ellipse if $e < 1$
>
> (b) a parabola if $e = 1$
>
> (c) a hyperbola if $e > 1$

It seems that few, if any, bodies would have truly parabolic orbits, since such a body would then have e precisely equal to unity. Bodies would most likely have either hyperbolic orbits (like single-pass comets, for instance), or they would have elliptical orbits. Any elliptical orbit would theoretically have some torus of eventual accessibility such as that shown in the earlier Figure 4. In theory, were there no inaccessible region about the Sun for a given body, that body would itself contact the Sun and be cleared, so there should always be some opening through the center of a planetary torus. Additionally, there must be some limiting size which would relate to the position in space where the gravitational attraction of the Sun is insufficient to draw the body back toward it. Lacking the limit to the orbital extent is described by the $e > 1$ case where "orbit" is lost, and the body would continue to move away from the Sun. It appears that there are many objects which are consumed by the Sun, and many which leave the solar system. These are the fascinating cometary bodies which do not conform to orbital planes near the Sun's extended equator, but rather have planes at truly random angles.

Admittedly, the model is likely to be oversimplified. Still, it's something to begin with. It does not require "astronomical expertise." The naiveté may be refreshing, and the result is an extension on the action learning project concerning Arrokoth [3]. One possibility is to compute the volume of the torus described for various planetary bodies. Then something could be said about the "clearing its path" notion. If there's anything redeeming to this effort, it necessarily includes some understanding of how volumes can be mathematically determined. In this case we have the marvelous theorem of Pappus which allows us to formulate

$$V = Ad = (\pi r^2)\, 2\pi R = 2\pi^2 r^2 R \tag{5}$$

where r is the radius of the cross-section of the torus and R is the mean radius of the torus, as indicated in Figure 4, above.

It is assumed that planetary objects have persisted for billions of years (or orbits), so that their paths (the interior of their tori) are swept clear. Perhaps the material encountered is absorbed by the planet or the material is ejected from the torus due to gravitational interaction.

This later possibility would be more active with larger bodies, so larger bodies may be more efficient at clearing their paths (or tori) than the smaller, Arrokoth-sized objects.

The gravitation disturbance factor is essentially what separates "planet" from "planetoid." This suggests that the tori of large planets might extend well beyond their perihelion and aphelion (r) distances. Furthermore, the tori may be more truly flattened, rather than circular. If this were to be considered, the r factor would require a more complicated expression. This might be warranted, however, since we have r^2 in the calculation of volume so that the error in r is exponentially increased.

Using available data for aphelion and perihelion distances for Earth, we find that

$$2r \approx 1.017 - 0.98327 \approx 0.033729$$

so that r is about 0.016865. Accepting this value for r, and using $R = 1$ for Earth, one obtains the result

$$V = 2\pi^2 r^2 R \approx 0.005611 \text{ AU}^3 \tag{6}$$

Since the volume of space within 1 AU of the Sun is $V = \frac{4}{3}\pi 1^3 \approx 4.18879$ AU^3, the result in (6) can be divided by 4.18879 to obtain 0.0013395. This latter number would seem to represent the approximate proportion of the full volume that Earth has access to in the course of its continued revolution about the Sun (the theoretical volume of Earth's orbital torus) to the volume of a sphere with a 1 AU radius.

The 0.005611 AU^3 found in (7) is "modest," since r is relatively small. It seems that, relative to the full volume of space at its distance from the Sun, the r for Arrokoth is similar. From published figures (including [3]), the aphelion and perihelion values for Arrokoth are about 46.442 and 42.721 AU, respectively. Using these values gives

$$2r \approx 46.442 - 42.721 = 3.721$$

so that r is very roughly 1.8605 AU. With R given as 44.5815, the toroidal volume is

$$V = 2\pi^2 r^2 R \approx 3046.097 \text{ AU}^3$$

The volume of a 44.5815 AU radius sphere is $V = \frac{4}{3}\pi 44.5815^{\ 3} \approx 371{,}152$ AU^3. Thus, the ratio of the toroidal volume to the full sphere's volume is about 0.0082071. This figure is similar to that found for Earth's ratio which was about 0.0013395. The similarity of these ratios suggests that the relative amount of obtainable space for "small e" bodies is approximately this value (of about 0.001 to 0.008).

It is known that there are many bodies with highly eccentric orbits. Such objects would have substantial deviations between their aphelia and perihelia, and therefore have correspondingly larger r values. Consider, for example, the Hale-Bopp comet which has an aphelion distance of 370.8 AU and a perihelion of 0.914 AU. For this body we get

$$2r \approx 370.8 - 0.914 = 369.886$$

giving r as 184.943 and a mean R of 185.857. The toroidal volume would then be computed as $V = 2\pi^2 r^2 R$ or about 1.25482×10^8 AU^3. The volume of a sphere with radius 185.857 AU is

$$V = \frac{4}{3}\pi\ 185.857^3 \approx 2.68921*10^7\ AU^3$$

So, the ratio of the toroidal volume to the full volume is about 4.66. In this extreme case, there is a toroidal volume exceeding the volume of the space in its corresponding mean distance sphere about the Sun. It is evident that there is only a small central region around the Sun excluded in the case of Hale-Bopp's torus. This would make the torus radius *r* very nearly the mean value of *R*. The findings are summarized in Table 1.

Table 1: Values of *r*, *R*, ellipse area, torus volume, and torus/sphere ratio for three solar objects.

Object	*r*: cross-section radius [AU]	*R*: torus radius [AU]	Ellipse area [AU^2]	V_t: torus volume [AU^3]	V_t / mean sphere
Earth	0.016865	1	3.141592	0.005611	0.0013395
Arrokoth	1.8605	44.5815	6249.286	3046.09726	0.0082071
Hale-Bopp	184.943	185.857	10,7454.76	1.25×108	4.666128

In attempting to create diagrams for the three bodies considered, it was observed that in the cases where *e* was near 0, the toroidal ring was barely discernable (at about 1/100th of a unit). In the highly eccentric case of Hale-Bopp, the torus volume overwhelms the mean distance sphere.

The theory may offer a suggestion as to how the portioning of the system's space is maintained. The planets with low eccentricities are evidently "stable." Those objects with eccentricities approaching one are "less stable," since they cover more space and may cross paths with other bodies.

There are some questions which arise. For example, can these notions also be applied to planetary moon systems? And can it be applied in some way to the orbit of the Sun about the galactic center? There's also the persistent question about the involvement of dark matter – the 80% of the matter in the Universe which remains unseen.

The findings imply that all bodies are in constant motion. The Sun itself has an orbit about the center of mass of the solar system. The radius of the Sun's orbit is only about half as much more than the radius of the Sun itself. It's hardly worth including the Sun's deviation from the center of mass in any calculation of planetary distances, but it is worthwhile to consider it as a point of theory.

The New Horizons team is congratulated for choosing to visit (if only briefly) the downgraded planet Pluto. Since it is outward bound and will continue to be so for the remainder of the 21st century, it looked like Pluto would simply be forgotten for most of this century. It is also believed that they selected an appropriate moniker for what was briefly called Ultima Thule. "Arrokoth" is a Powhattan word suggesting "sky." While Arrokoth may still be the most distant object visited by a spacecraft, the New Horizon's mission is ongoing. There will be a new "ultimate target" (for that moment). The spacecraft traveled seven billion km to peek at Arrokoth on January 1, 2019. It will undoubtedly tally up a few billion more before it goes silent.

This discussion demonstrates that action learning can be contagious. Project work, like [3], can serve to inspire readers to consider these and similar matters. It is also an opportunity to work with ellipses and tori, and to consider how these occur in the solar system. There are observations concerning symmetry which can be made concerning the theoretical ellipse. For one thing, as evidenced earlier at equations (2) and (3), the ellipse has mirror symmetry across both the major and minor axes. This results in four similar curves, making the evaluation of area within a quadrant the pragmatic way to perform definite integration as in [3]. The fact that the cosine function is even, for instance, is verified visually by plotting $y = \cos x$ and noting that the curve is mirrored across the y-axis. This is an instance of action learning for, as was noted at the outset, symmetry is a recurring concept throughout one's academic life. There are notable applications in design where two-dimensional symmetries are evoked. Even where three dimensions are active, it is pragmatic to model certain celestial motions in a plane. Visually, we can work solely with the information provided in Figure 5 for Arrokoth. Notice that only a single quadrant need be depicted. In the Arrokoth case, the toroidal width might be only slightly wider than the physical curve (denoted "UT"). On this scale, Earth stays close to the lower-left asterisk denoting the Sun (or second focus). The tori of both Earth and Arrokoth might roughly have the thickness of a Hula-hoop in comparison with the highly eccentric Hale-Bopp comet, which might more closely resemble an over-inflated truck inner tube. Figure 5 shows only the elliptical component of Arrokoth. We can judiciously select quadrant I (QI) in Figure 5.

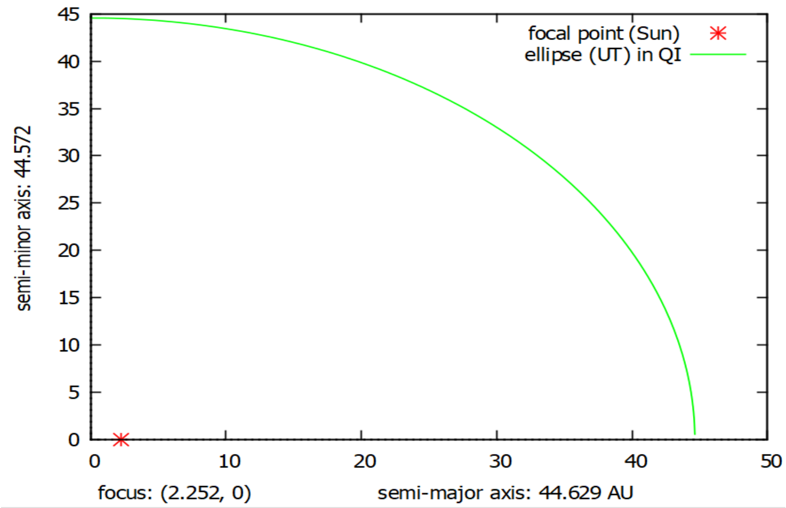

Figure 5. Illustration of Arrokoth's (UT's) orbit. Earth's orbit (not shown) is 1 AU.

It would be remiss not to mention that higher order symmetries can be considered. Clearly there are analogous planar reflections in three-space. As Coxeter [4] explains, there are symmetries in higher dimensions as well. While humankind is relegated to three spatial Euclidean dimensions, there is no particular reason why there can't be more dimensions "beyond human perception." While it might seem to be a rather unnecessary assumption, more dimensions may offer a potential place to find dark matter, for instance. This would likely require innovative (Hamiltonian-style) mathematics to handle 4-space calculations. There has been mathematical success in describing 4-space rotations. See, for example, Mortari's *On the Rigid Rotation Concept in n-Dimensional Spaces* [11].

Discussion

It is understood that there are so called secular components to the orbits of solar system objects. There are pedagogical reasons for wanting to know why the orbital pattern of this object might have "analytic continuation" (as pondered in [3]) for example. From the experience of actively computing orbital area and toroidal volumes, one may develop the confidence to consider properties beyond those of area and volume. According to Kepler's Law, planetary orbits move in planes. These planes all have some deviation (inclination) from the Sun's orbit, so these planes are merely approximate models for motion which is truly three-dimensional.

While the planets of our solar system deviate only slightly from lying on a common plane, there are myriad smaller objects (primarily comets) which freely take on "randomly" oriented orbital planes. It appears that the Milky Way has an overall planar structure, as well. Most galaxies tend to appear situated in planes at various angles. The suggestion is that as material rotates, it tends to thin out. The rings of Saturn might be an example of this, since they are clearly very thin.

At any rate, we can certainly consider these more technical elements of motion with a goal of understanding more fully how bodies in orbit truly move. It is of interest that while the isolated motion of *two* bodies seems to be understood quite well, there are complications as soon as even *three* bodies interact. We know that no model will be perfect (since multiple bodies will always be present), and our mathematical models of planetary orbits can only approximate the true physical situation.

Figure 6 is a composition of images which presumably provides a three-dimensional image of the object when viewed with those green and red 3D glasses. It is curious that this can be done (just as it is with various pairs of early Apollo images taken from slightly different angles). There is justification to this, since it may allow the viewer to get a sense of depth. It should remind us that space has precisely three distinct axes of Cartesian direction. It is pragmatic to employ a two-dimensional model, but there is nothing in the Universe which is truly two-dimensional (nothing *seen* by us, at least). There is some non-zero depth to this page, for instance.

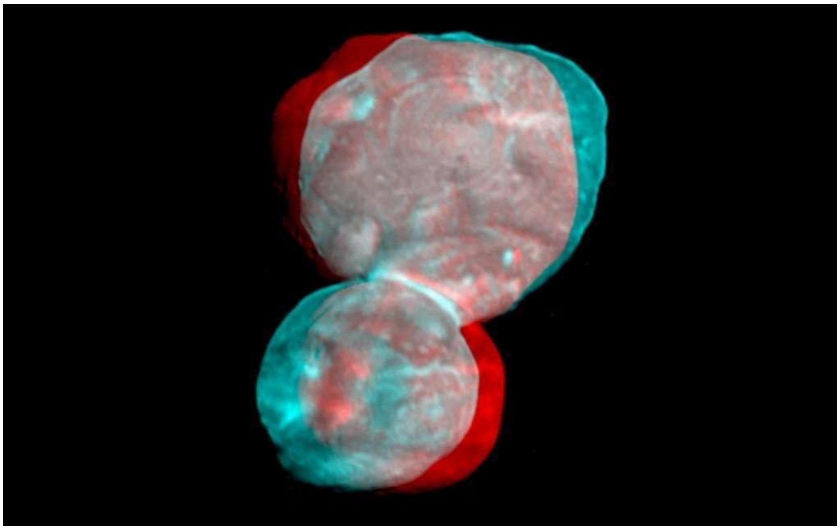

Figure 6. A composite "3D" image of *2014MU₆₉*.

The object in Figure 6 can be viewed as a composition of two bodies. For a time in 2019, the combined object was called "Ultima Thule" (Latin for "farthermost Thule") [19]. Until New Horizons viewed $2014MU_{69}$ (which was an even earlier name for Arrokoth), it wasn't known whether it was a single object or multiple objects in orbit about one another. Since there is a solid connection, a single object (only about 35 km long) is now recognized as being Arrokoth.

As of November 2019, the name "Arrokoth" has been accepted as the official name of this object. Arrokoth is a more appropriate name, since New Horizons has not completed its mission, and since it is now known to be a single object. Specifically, the object is designated "486958 Arrokoth" [10]. It seems that the new name for Ultima Thule was appropriately chosen. The "bear" has seen over that mountain now, so it's off to the next mountain. There's always more to see. As noted earlier, "Arrokoth" is suggestive of "sky" in Powhattan, and the sky is *not* a limit.

Conclusions

The area within the ellipse of Arrokoth (the KBO 2014MU$_{69}$) was calculated. In this simple ellipse model, it is pragmatic to first find $f = ea$. Multiplying the ellipse eccentricity (0.05046617) by the semi-major axis (44.629) gives a focal distance of 2.2522861 Astronomical Units (AU). As Tauber illustrates, the value for b can be found from $b^2 = 44.629^2 - f^2$ so that $b \approx 44.572$ AU [16]. The focal distance f is the distance from a focus to the *center* (not the distance *between* foci, which is often taken to be "focal length").

Arrokoth is a new object of study, so there is more to learn. The results evidently show that Kepler's First Law for planetary orbits can be used when finding the area within an ellipse. The process used is also useful for calculating other areas of orbits. This is naturally extended to volumes in three-space.

It is apparent that the orbits of distant objects aren't known empirically. For instance, the orbit of Pluto (discovered in 1930) has only been observed for 90 years. Consequently, less than one-third of its orbit has been observed. Since at least a quarter of an elliptical orbit is needed to fully describe it, we've only been able to verify the orbit of Pluto visually very recently. Together with the other elements of motion, there is still some uncertainly about how this object orbits the Sun.

In the case of Arrokoth, which is believed to have an orbital period like that of Pluto (of about 300 years), we currently have less than six years of observational data, since it was only discovered in June of 2014. An interesting problem for further action learning might be to resolve an orbit from four observed locations. It would seem that at least four points would be needed without additional information concerning eccentricity and inclination. Action learning can therefore be *further* extended in considering such problems.

The concept of "measurement" draws on related notions of the usefulness of models despite the ever-present imperfections in modeling any real situation. In the case of Earth (or Arrokoth), we have a prototype which naturally cannot be fully modeled (without replicating the totality of matter in the solar system, etc.). All models are consequently lacking some information. Never-the-less we can use models, with the caveat that what is depicted via the model is but an approximation of the actual physical situation. In the future, there may be a model which includes the presence of dark matter, for instance. At that time our model may be "better," however it would still be deficient.

This exposition demonstrated that there are generally further physical and mathematical implications to any study. This is what makes the UJMM [18] an extensive resource for future action learning. As was noted, it is likely that different readers will be inspired by a write-up in UJMM in different ways and to varying degrees, a single project may therefore offer multiple extended Project possibilities.

In addition to the variations in topic directions which a single project might elicit, there is the synthesis of mathematical ideas that go into project work which is of importance. This is what makes the Calculus sequence so interesting. Calculus seems to draw upon virtually all of the concepts of mathematics we cobble together across the educational spectrum (K-16+). In addition to the ever-present symmetry, concepts of counting, measuring, patterns, recurrence relations, representations, approximation, inequalities, probability, and many famous problems are all seen in the Calculus sequence. It's perhaps the notion of Famous problems which is most compelling. Consideration of classical mathematics problems has been the force driving its continued development. The well-known ancient problem of squaring the circle, for instance, led to the advanced concept of transcendental numbers. It took centuries of thinking about the squaring the circle problem to ultimately find that it is unsolvable when restricted to algebraic numbers, however it's trivial ("as easy as π," one might say) in the extended set of transcendentals.

Relative to [3] are the ongoing problems of planetary distribution and the universal distribution of matter. The work of Vera Rubin and others has led to the conclusion that about four-fifths of the Universe is currently unseen. What we collect by detecting electromagnetic radiation is figuratively only the tip of the "cosmic iceberg." It will require further study to unlock this mystery. Our present efforts to develop working models for a "string" theory may be work in the right direction. Additionally, there are concepts of "knotting" and "twisting" which may be bound up in the issue of perceiving matter in space.

It is of historical interest that Kepler [7] believed that he had resolved the matter of planetary distribution in the solar system. As revealed in his twice-published *Mysterium Cosmographicum*, Kepler found that the distances between planetary spheres (or orbital paths) were precisely as would be found by enclosing the five platonic solids in concentric spheres. There is the recognizable image, shown in Figure 7, which illustrated this 400-year old theory. Kepler was quite certain that his findings concerning inter-planetary spacing were factual. He may have been emboldened by his success with the Laws of Motion in proposing the Platonic solar system model.

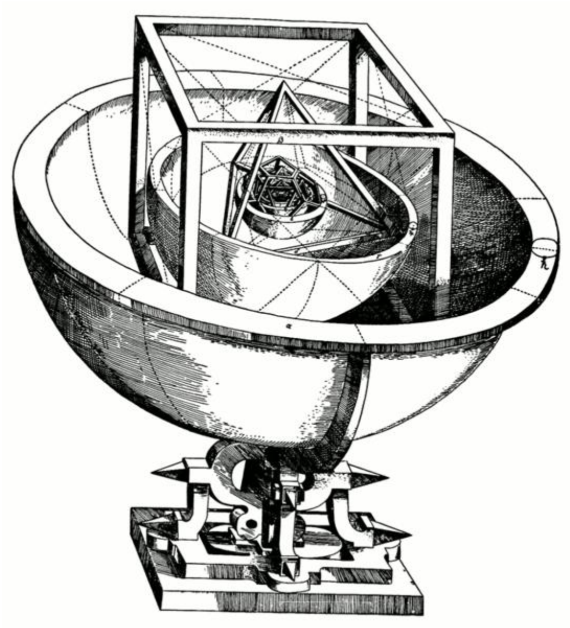

Figure 7. Kepler's Platonic solid model of the Solar System

While it has since been found that Kepler's model is flawed, one can't help but sense that there is indeed something "golden" about the interplanetary spacing evident in the solar system. This is what endures with Kepler's work. We desire order and regularity to things like the motions of heavenly bodies. We also like to think that the harmony out there is replicated in our earthbound endeavors such as writing songs and doing calculations. This is that "holy grail" of science -- to reduce all of mind and matter to a single unifying theory. That finding would at least pragmatically be efficient (and it might connect us to some "cosmic harmony," as Kepler proposed). We can sympathize with Kepler in his efforts to find music in the cosmos. He suggests that our music is the same as that to be heard throughout the cosmos.

Before concluding, it is worth noting that it is diurnal motion which is undeniably humanity's earliest unit of time. This is part of the song as well, and with that music comes all the issues of angular momentum and torque which one might attach to any robust study of dynamic systems. In higher mathematics the complication is often in modelling these dynamic systems. Recent findings (for example, those of Schrödinger as described by Gribbin [6]) suggest that there are limits to what can be determined using mathematics when mass and motion are both involved.

This limitation concept is included in the work of Church and Turing [17]. This is one of the many directions new investigations are following. We like to know when a problem has no solution. It may be enough to even say that a problem has no quick (polynomial time) solution. An important example concerns the inability to quickly factor large composite numbers. It is believed that no such fast factorization algorithm exists (and our security depends on that being true). Never-the-less it is still an open question.

As a final note to this digression, there should remain a direct tie of human understanding as machines continue to handle calculations with greater speed. There were thoughts about mechanical computation many decades before we really had a viable electronic computer. "Computability" was, until only recently, theoretical. It continues to be a real and growing field of study (and may now remain in symbiosis with the computer). Evidently, there may evolve a "quantum" way of computing as described in the works of Nielsen and Chuang [13] and others.

It is unlikely that in "traditional mathematics" a fast factoring algorithm is possible, as Shor [14] contends. However, one *might* find a means of quickly determining the factors of a large composite number, if one had qubits to work with. ("Qubit" is the accepted name for "quantum bit," and is pronounced as is the word for the Ancient Egyptian standard length, "cubit.") This supports Lobachevsky's conjecture that all of mathematics will ultimately have applications. We need only create the applications. Governments and businesses are investing in the future of the emerging quantum computing paradigm.

The digressions suggested above collectively show that anyone wishing to compose a mathematics project can include observations which might supply greater overall reader interest. It should be mentioned that a project composer has an opportunity to share ideas (as was done with [3] in [18]). The ideas can usually go farther. We would like to deny any "ultimate Thule" as regards mathematical ideas. Now let's see what there is to see beyond Arrokoth.

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