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THE POSSIBILITIES OF DEVELOPING ALGEBRAIC THINKING IN THE MIDDLE SCHOOL UPPER GRADES

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Abstract. Some elements of algebra are a focal point in mathematics education in upper middle school grades. Many researchers in mathematics education promote an idea that algebraic reasoning should be stronger integrated in mentioned grade levels. This article focuses on algebraic contents in mathematics curricula of upper grades in middle school in Bosnia and Herzegovina, Croatia, Macedonia, Montenegro, Serbia and Turkey in the order to estimate what elements of algebraic thinking they enabled by these curricula. By comparing mathematics teaching programs in these sociopolitical communities for 7th and 8th in middle school grades, the authors analyze the components of algebraic thinking that are provided by so designed curricula.

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Key words and phrases: comparing of mathematics curricula**,** algebra, Bosnia and Herzegovina, Croatia, Macedonia, Montenegro, Serbia, Turkey, algebraic thinking, basic algebraic ideas in teaching and learning mathematics.

1. Introduction

Our experience in mathematics classroom, as well as our reflections on the experience, suggest that a much greater opportunity to influence the development of students' mathematical proficiencies has a better and consistent design of the mathematical curriculum than the teachers themselves in that classroom. Those who influence and those who design mathematical curricula can incorporate in them both their principled-philosophical orientations, the intentions and goals they wish to achieve by designing those curricula. The authors are convinced that there should be a constant and continuous dialogue on the quality of mathematics education between mathematics teachers, on the one hand, and the academic and social community, on the other. In their efforts to contribute to these negotiations, this group of mathematics education researchers focused on mathematics curricula in middle school (see: [8], [19]).

Mathematics curricula for upper grades in middle schools should, inter alia, aim to develop algebraic thinking among students who are learning. In order to gain insight of these possibilities, we examined mathematical programs for 7th and 8th middle schools grades in Bosnia and Herzegovina [B&H = RS + FB&H][1], Croatia [CRO], Macedonia [MK], Montenegro [MNE], Serbia [SRB] and Turkey [TR], with the intent to reconstruct the elements of algebraic thinking, in accordance with the Shelley Kriegler classification [26] and [36], which these curricula allow.

Susan Jo Russell once [39] estimated that the efforts of mathematics teachers in a mathematical classroom should be an incentive to algebraic thinking of students. Typically, in school systems, mathematics curriculum designers divide areas that allow for the development of arithmetic thinking from areas that support the development of algebraic thinking. A broader conception of algebra emphasizes the development of algebraic thinking, rather than just the skilled use of algebraic procedures (known under the name: 'fluency of procedural proficiencies' [23]. Although algebra is not the main focus in the mathematics curriculum in middle school, students of upper grades should be prepared to be familiar with algebraic generalization and especially with some of algebraic abstraction. The authors are convinced that these students should master other mathematical skills according to Kilpatrick at al. Classification [23]: 'conceptual understanding', 'strategic competence' and 'adaptive reasoning'. The publications on middle school upper grades students' algebraic thinking has not yet generated a framework for systematically characterizing students' algebraic thinking, for example, based on the cognitive learning theory such as the SOLO [7] or MATH taxonomy [40] and [3].

In the essence of things, if there is a theoretical framework for the development of the students' algebraic thinking, then this framework could be incorporated into the mathematical curricula of the upper classes of middle schools. Of course, there is a considerable number of studies of students' algebraic thinking in upper grades in the middle school (for example: [6], [13], [15], [17], [22], [24], [25], [28], [32], [34] and [41]). Most of these articles are devoted to describing the elements of algebraic thinking in such students.

In this article, the authors will compare the algebraic content in these curricula, comparing mathematical curricula earlier for the 7th and 8th grades of the Middle School of the aforementioned sociopolitical communities. At the same time, we will evaluate which possibilities of the development of a student's algebraic thinking open up such designed curricula. Naturally, in these activities, we will be able to follow to the analogous activities and experiences of the academic community of researchers in mathematics education (for example: [8], [12], [20], [33], [19] and [44]). The considerations outlined in this paper should, in our biew, be a strong support for the middle school community of mathematics teachers in negotiating with the community about increasing efforts and (if possible) material means in improving the quality of mathematics education in these schools. These beleifs and commitments can appropriately persuade some policy makers and educational leaders to institute or support careful consideration of enactmented curriculum policy. For example, for mathematics education standards to be formed, understood and enacted, teachers, policy makers, and leaders need to collaboratively examine and clarify their

[1] Bosnia and Herzegovina is a very specific political creation. It consists of two entities: the Republika Srpska [RS] and the Federation of Bosnia and Herzegovina {FB&H]. The entities authonomusly establish their curricula for all levels of school systems.

interpretations and vision for enactments of the official mathematical curriculum (compare with [9], [31] or [35]). It's not enough to adopt someone else's standards, as is the common practice, participants at all stakeholders need to operationalize what it means to take the standards in the mathematics classroom, ensure that interpretations of the standards are understood, and investigate the extent to which the type of curriculum defined for use is designed to support these interpretations of the standards.

2. Algebraic thinking

Every reader of texts covering the domain 'Research in Mathematical Education' should always bear in mind that 'Algebra', 'school algebra' and 'the pedagogy for the teaching of school algebra' are very different areas, although they are very close [37].

When does algebra begin? Algebraic thinking begins to emerge among the students learning arithmetic when they begin to ask questions and seek answers to them. For example, in the second half of the first / second grade students learn the concept of even numbers. This is realized by observing and remembering strategies for adding the same numbers to 20:

$$1 + 1 = 2,\ 2 + 2 = 4,\ 3 + 3 = 6,\ \dots,\ 9 + 9 = 18,\ 10 + 10 = 20.$$

As it is customary, students look at these equalities from left to the right, and they remember them. But the teacher can ask them to describe what they see if they look at these equalities from the right side to the left side: Number 2 is written in the form as the sum of two identical numbers, number 4 is written in the form as the sum of two identical numbers, and so on. When a teacher asks for a conclusion, it is possible for the students to say: 'Some natural numbers can be written / presented as a sum of two identical natural numbers (and some cannot)'. Researchers in mathematics education are very interested how students would respond to the following teacher's request: 'How will we register it?' This is a convenient moment when a teacher can ask from students to accept that this equality is not an identity.

We use this opportunity to remind the reader of this paper that 'equality' is a predicate of the second order that satisfies the following axioms:

$(\forall x)(x = x)$,

$(\forall x)(\forall y)(x = y \Rightarrow y = x)$,

$(\forall x)(\forall y)(\forall z)((x = y \wedge y = z) \Rightarrow x = z)$,

$(\forall x)(\forall y)(x = y \Rightarrow f(x) = f(y))$, where f is any functional symbol of length 1[2],

$(\forall x)(\forall y)((P(x) \wedge x = y) \Rightarrow P(y))$, where P had any predicate symbol of length 1[3].

Also, here we want to remind the reader on the fact that whenever we talk about elements of logical thinking, we always think of 'the logic with the equality'. In addition, in analyzing elements of algebraic thinking, we will use quantifiers as categorical terms. These are the slogans or compound words by which we are trying to describe if any statement applies to 'all objects of

[2] It is common in Mathematical logic for a functional symbol f is said that 'f has length 1' if it acts on only one variable.

[3] It is common in Mathematical logic for a predicate symbol P is said that 'P has length 1' if it acts on only one variable.

the observed set', or, refers to 'some objects of the observed set'. This is about quantifiers: the quantifier of universality and the qualifier of existence. In this, it is important to take into account the domain to which quantifiers are applied. If **Q**, and **R** denote the field of rational numbers, and the field of real numbers, respectively, then the formula $(\exists x)(x^2 = 2)$ is interpreted in the domains **Q** and **R** has different validity: In **R** this is a valid formula but in **Q** it is not true, because there is no element x in **Q** such that $x^2 = 2$. To show that this formula is not a valid formula, it is necessary to reference the following logical tools: 'the law of contraposition' and 'the double negation law'. Therefore, apart from 'the modes ponens' and 'the rules of universal generalization' as the rules for concluding, students of the upper grades in the middle school must understand and know how to use the two logical principles mentioned above.

Maria L. Blanton and James J. Kaput [5] explain that algebraic thinking can be seen as a process in which the generalizations of mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal ways. A considerable number of researchers were interested in the phenomenon of algebraic thinking among primary school pupils ([2], [6], [13], [15], [17], [22], [34], [36] and [37]). Algebraic thinking, in this paper, followed by Shelley Kriegler [26], is organized in two main components:

(1) The development of mathematical thinking tools and

(2) The study of fundamental algebraic ideas.

Mathematical thinking tools are analytical habits of mind. They are organized around three topics: problem- solving skills, representation skills, and quantitative reasoning skills. Fundamental algebraic ideas represent the content domain in which mathematical thinking tools develop. They are explored through three lenses: algebra as generalized arithmetic, algebra as a language, and algebra as a tool for functions and mathematical modeling.

Mathematical Thinking Tools	**Informal Algebraic Ideas**
Problem-solving skills • *Using problem- solving strategies* • *Exploring multiple approaches/multiple solutions* Representation skills • *Displaying relationships visually, symbolically, numerically, verbally* • *Translating among different representations* • *Interpreting information within representations* Reasoning skills • *Inductive reasoning* • *Deductive reasoning*	Algebra as abstract arithmetic • *Conceptually based computational strategies* • *Ratio and proportion* Algebra as the language of mathematics • *Meaning of variables and variable expressions* • *Meaning of solutions* • *Understanding and using properties of the number system* • *Reading, writing, manipulating numbers and symbols using algebraic conventions* • *Using equivalent symbolic representations to manipulate formulas, expressions, equations, inequalities* Algebra as a tool to study functions and mathematical modeling • *Seeking, expressing, generalizing patterns and rules in real-world contexts*

	• *Representing mathematical ideas using equations, tables, graphs, or words* • *Working with input/output patterns* • *Developing coordinate graphing skills*

Figure 1: *Components of algebraic thinking* (Taken from [26])

Michael Yerushalmy [45] support the orientation that the concept of functions is one of the fundamental subjects of algebra and that it ought to be present in a variety of representations in the teaching and learning of algebra from the very start. Some researchers believe that the concept of function is one of the central ideas of pure and applied mathematics and many of them suggest that teaching algebra should be based on functions (see, for example, [29]). The coherence of the observed mathematics curricula was achieved, among other things, through central concepts, such as function, equality and inequality. Through the observation of linear functions, linear equations and linear inequalities with one unknown, the designers of the observed mathematical curricula allow mathematics teachers in their classroom to stimulate the development of components of algebraic thinking. The functional approach to algebra that enables for each of the analyzed curricula in the upper grades of middle school to be applied, has given the chance to students to solve problems with different approaches and strategies using different representations of a function such as the graph, the table or the letter-symbolic (analytic) form and not only with the formulation and solution of equations.

Algorithms that enable the finding of the solution of linear equations and inequalities strongly stimulate the development of the component of algebraic thinking in the Middle School students. Students should be able to recognize, understand, accept, and be able to use categorical terms such as: equation, inequality, and a system of equations. Also, these algorithms help the development of analytical awareness among these students. An equation or a system of equations may not have the solution at all, or to have the unique solution, or to have more than one solution. Understanding these differences as well as the conditions under which these varieties are realized, we classify as analytical components of consciousness.

Of course, this paper is significantly different from the following papers [32], [17], [22], [24], [25] and [34].

It is customary to expect that middle school students of the 7^{th} and 8^{th} grades understand, accept, and use algebraic symbolism to a significant extent. Also, the orientation involved in the analyzed mathematical curricula mean, among other things, that the students of the upper grades of middle school are capable of recognizing, understanding and accepting the processes of inductive (to the fullest extent) and deductive reasoning.

3. Comparison of the Curriculum models

In the domain of international research and analysis of mathematical education systems, it is estimated that mathematical curricula are the most universal paradigm under the strong focus of many researchers. We deeply believe that when approaching and designing mathematical curricula, one socio-political community implicitly expands its principled-philosophical orientations and its relation to mathematical education. We, too, believe that policy makers in education and designers of mathematics curricula incorporate in them their political commitments related to education including their ethical orientation. This last statement is especially related to the selection and accuracy of the goals of teaching mathematics, the description of teaching tasks which enable achievement of the chosen goals, as well as the list of desirable outcomes of such mathematics education.

Our intention in writing this paper, was based on the belief that comparing the teaching contents in the mathematical curricula of the observed socio-political communities helps to make observations of the goals of teaching mathematics in the domain of algebra and also helps evaluate the possibility of developing a student's algebraic thinking that these contents allow.

Our intention with this paper is to open a dialogue between the designers of mathematical syllabi. In the school systems of Bosnia and Herzegovina, Croatia, Macedonia, Montenegro, Serbia and Turkey, the term 'teaching plan' refers to the planned annual number of hours of mathematics. The term 'teaching program' refers to the objectives of teaching mathematics (general and individual), teaching contents, planned outcomes of mathematics and didactic instruction for teaching realizes. In Bosni and Herzegovina, Macedonia and Montenegro, the basic education system (primary grades and middle school grades) lasts nine years. Therefore, the corresponding grades are: 8^{th} + 9^{th} (B&H, MK and MNE) and 7^{th} + 8^{th} (CRO, SRB and TR).

Information on mathematics teaching plans for the observed grades in middle schools in the mentioned countries is presented in the following table:

Table 1: Teaching plan of mathematics for 7^{th} / 8^{th} grade

State	Number of Weekly Classes	Number of Annual Classes	FB&H = 1	Grades
B&H [FB&H]	4	140	1.000	VIII
	4	132	0.923	+IX
B&H [RS]	4	144	1.029	VIII
	4	136	0.971	+IX
Croatia	4	140	1.000	VII
	4	132		+VIII
Macedonia	4	160	1.143	VIII
	4	160	1.143	+IX
Montenegro	4	136	0.971	VIII
	4	128		+IX
Serbia	4	144	1.029	VII
	4	136	0.971	+VIII
Turkey	5	180	1.286	VII
	5	180	1.286	+ VIII

Legend: As a unit of comparison, we have taken the annual number of mathematics hours in FB&H

The teaching contents of mathematics for the observed grade are the following

Table 2: Algebraic contents in the curricula for 7th and 8th in the middle school grades

Thematic content	**B&H** RS	**B&H** FB&H	**[CRO]**	**[MK]**	**[MNE]**	**[SRB]**	**[TR]**
Proportion and percentage account		+ (20)	+		+ (9)	+	+ (35)
Squaring, rooting and powering			+	+	+ (18)		+ (25)
Real Numbers,	+ (15)		+				
Coordinate system in level and function of direct proportionality		+ (20)	+	+	+ (18)	+	
Algebraic fractions	+ (42)		+			+	+ (25)
Monomials and polynomials		+(30)		+		+ (46)	+ (10)
Some fundamental functions	+ (18)		+				
Linear equations and inequalities with one unknown	+ (26)	+ (30)		+	+ (25)	+ (18)	+ (40)
Linear function	+ (18)		+	+	+ (24)	+ (12)	
Systems of two equations with two unknowns	+ (19)	+ (20)	+	+	+ (20)	+ (12)	

Legend: Numbers in brackets indicate the suggestions of the Ministry of Education about the necessary hours for teaching and learning of the noted themes. With the '+' mark we indicate that the relevant topic is in the observed curriculum, although we do not have reliable information on how many school units are planned for this content.

[B&H / RS] The Algebraic Content in the Republic of Srpska (the Entity of B&H) 8th grade in middle school are: Real numbers (15 hours), Algebraic fractions (42 hours) including monomials and polynomials, some fundamental functions (18) including coordinate system. In the 9th grade, the algebraic contents are: linear equations and inequalities (26 hours), linear functions (18 hours) and systems of two equations with two unknowns (19 hours).

[B&H / FB&H] The algebraic contents in the math courses for the 8th and the 9th grade of the middle school in the Federation of B&H are the following: Proportion and percentage calculus, coordinate system in level and function of direct proportionality, monomials and polynomials, linear equations and inequalities with one unknown, and systems of two equations with two unknowns.

[SRB] In Serbia, in the 7th grade, direct and inverse proportionality is taught learns together with dependent variables and their graphical representation in the coordinate plane. As part of this topic, there are proportions and percentages, with the students becaming familiar with these terms in the 5th and 6th grades. In the 5th grade, they are introduced to the concept of scale, in more detail, the percentage is in the 6th grade, and in the 7th grade the whole topic is completed and the application is done in detail. Squaring and finding roots are taught in the 8th grade, and within this topic, the terms irrational and real numbers are introduced, so real numbers are also included. Later, the degree is studied within the teaching theme monomials and polynomials, which represents the most important topic in the 7th grade.

[CRO] Mathematical curricula for the 7th and 8th grades of the middle school in Croatia are designed very specifically and significantly differ from others with which we compare them. According to the official documents of the Ministry of Education of the Republic of Croatia, in the 7th and 8th grades in the middle school, the algebraic contents within the subject of Mathematics are as follows: coordinate system, proportions and percentage calculus, system of linear equations with two unknowns, monomials and polynomials, squaring, square roots and powers, real numbers.

[MK] Mathematics programs for the 7th and 8th grade of the middle school in the Republic of North Macedonia have been designed significantly more adapted to the teachers than is usual with the other curricula that we observe. The specific objectives of teaching mathematics are emphasized for each planned individual topic. The arithmetic-early-algebraic and algebraic contents in the 7th and 8th grade in the middle school in the Republic of Macedonia are as follows: integers, decimal numbers and fractions; monomials and polynomials, algebraic fractions; some special functions such as sequences; linear equations and inequalities and coordinate systems.

[TR] The teaching plan of mathematics for the 7th and 8th grade in the Middle School in the Republic of Turkey is very extensive (180 hours). The algebraic content included in the teaching programs are: proportion and percentage calculus, Squaring, taking square roots and powers, algebraic fractions, monomials and polynomials and linear equations and inequalities.

Although at the first glance, it appears that the list of topics taught in various countries has random capitalization, this is not the case. This selection of topics is in the long-standing and deeply rooted tradition of school systems in observed societies. The authors of this paper are very close to a commitment that significant equalization could be found for a perfectly acceptable justification. One consideration of the possible unification of algebraic contents was offered by Ángel Alsina in his recently published paper [1].

4. Interaction between the design and use of curriculum materials

By achieving students' mathematical knowledge, skills and proficiencies, teachers try to base their classroom activities based on recognized, assessed, and previously acquired mathematical knowledge, skills and proficiencies of their students in accordance with the current annual plans of realization of teaching with almost strict respect of school mathematics curricula. Teachers should plan their school activities so that the largest number of their students reach the planned goals of teaching mathematics at a higher level. Admitting the belief that the goals of teaching mathematics are achieved in the observed grades by choosing appropriate teaching assignments, the conclusion is that qualitative teaching of mathematics depends essentially on the quality of teacher education and the quality of school mathematical curricula.

Our intention in this paper is to evaluate the possibility of developing algebraic thinking among students in the final grades of the middle school in the observed socio-political communities based on the analysis of correspondent curent mathematical curricula. Although we will not engage in an analysis of the education of mathematics teachers in the Middle School in the observed social communities, we will nevertheless point out the link between the necessary knowledge of such teachers and school mathematical curricula. We are close to the point that desirable components of knowledge, skills, and proficiencies for Middle School mathematics teachers can be described by means of the following categories:

(a) School mathematics [SM];

(b) Mathematical knowledge necessary for the realization of teaching [MK];

(c) Methodological knowledge necessary for the realization of teaching [PCK] and

(d) Understand the process of teaching and student learning mathematics at an appropriate level of maturity [UPTL].

In several research papers (for example: [4], [9], [13], [15]), their authors' beliefs about the categories of necessary mathematical knowledge of Middle School teachers are presented on the following way:

Math1: Knowledge of topics [KoT];

Math2: Knowledge of the structute of mathematics [KSM]; and

Math3: Knowledge in mathematical practice [KPM].

Also, in the above mentioned articles, the methodical knowledge necessary for the realization of the Middle School of teaching mathematics includes the following sub-categories:

Knowledge mathematics teaching [KMT];

Knowledge of learning mathematics [KFLM]; and

Knowledge in Mathematical Learning Standards [KMLS].

These two groups of sub-categories link the binding components that we recognize as the affective goals of teaching mathematics. For example, they are knowledge, understanding and acceptance that we identify as desirable development of components of various forms of mathematical and logical thinking among students.

This is the place and moment in our analysis when a fully justified question can be posed: Do we know to what extent the middle scholl teachers of mathematics in the selected social communities are familiar with these mentioned knowledge, skills, abilities?[4] Although our intention in this article is not to evaluate the level of quality of mathematics and methodical teacher in service education, we will nevertheless point out that the responsibility for this education is to be deployed between the Ministry of Education, the University Administration Authority, the State Pedagogical Institutes and the management boards of school systems. More precisely, it is quite justifiable to consider whether middle school teachers in the mentioned societies are:

- able to recognize the components of algebraic thinking,
- able to choose teaching tools through by which these components are encouraged to develop, and
- whether they are qualified to assess how successful they are.

Even if affirmative answers to previous considerations are not expected, it is entirely justifiable to expect that the official mathematical curriculum and teaching materials that should accompany it are designed so that Middle School teachers in service can find instructions for almost all questions and concerns with whom they meet in their teaching practice and in their efforts to realize this practice professionally.

In the literature (for example, in [38]), one can find the researcher's belief in mathematics education that one of the important support in the efforts to raise the quality of mathematics education in the Middle School is the appropriate teaching's system of activities. This system of

[4] To a perfectly justified question: "Do we know the extent to which Middle Scholl teachers of mathematics in selected social communities are familiar with these mentioned knowledge, skills, abilities? " unfortunately, we do not have a positive answer.

classroom activities should be consistent with a mathematical curriculum for the appropriate grades [31]. A growing body of evidence indicates that characteristics of mathematics curriculum have strong impact on teaching and students learning (e.g., [31]). Indeed, some of them (e.g., [30]) are convinced that teachers' principal-philosophical orientations were consistent with the type of curriculum they used in concrete cases. Lately, the term 'curriculum ergonomics' (e.g., [11], [33], [43]) appears in the literature, which is interpreted as "exploring the interaction between the design and use of curriculum materials" [43]. This includes, among others, the design of the curriculum, the necessary accompanying materials for teaching [33] and didactic instructions.

5. Conclusion

By perceiving and comparing algebraic contents in the curricula of the middle school in B&H, Croatia, Macedonia, Montenegro and Turkey, we are very close to concluding[5] that these curricula allow for the development of many components of algebraic thinking among students of these schools. Thus, the makers of educational policies and designers of these observed mathematical curricula have created social environments that should enable the development of algebraic thinking among middle school students. A much more important part in these endeavors is the quality of mathematical education and high-quality didactic education of middle school mathematics teachers. These mathematical curricula should be followed by additional detailed instructions with a list of goals of mathematics teaching in the upper grades of the middle school. Also, these additional instructions should contain a detailed description of the teaching tasks that enable the achievement of the planned teaching objectives. At the end of these considerations, the authors express their opinion as a professional orientation: in each of the observed sociopolitical communities a body to monitor, analyze and suggest improvements in mathematical curricula should be formed.

Based on our study results, we are inclined to suggest ([21], [18], [46]) that the focus of observation of the middle school of mathematical education should shift from students and teachers to education policy makers, designers of mathematical curricula and the board of management of state pedagogical institutes as responsible institutions with significantly more impacts on raising the quality of this mathematical education. Also, we are very inclined to believe [27] that the quality of mathematical education, in general, and thus the development of algebraic thinking in middle school students, would be much more present if the curriculum, the accompanying material and the corresponding didactic instructions were prepared in accordance with the results of modern scientific thinking.

We identify these two commitments for a number of reasons. For illustration purposes, one of them is that in some of the observed socio-political communities (for example, B&H, MK and MNE) there are no documents that could be identified as National Educational Standards in mathematics education. The long-standing experience in research activities within the domain 'Research in Mathematical Education' of the fourth author of this paper gives us a justification of what we accept with a high degree of restraint of official information on the success of mathematics education in the middle school systems of the observation centers.

Let's do one thought analysis:

[5] What does assesment “very close to concluding” means? Is there another step? Are we uncomfortable making the final decision? We are convinced that more research should be done before the final decision is made. In that case, the conclusion could be formed as a proposal for future action.

The mathematics curriculum is delivered by the Minister of Education at the suggestion of the Pedagogical Institute. The Pedagogical Institute designs the mathematics curriculum at the suggestion of an ad hoc group of mathematics teachers. Mathematics teachers in the middle school system realize the process of teaching students. This realization is overseen by the school counselor, school manager and counselors of the Pedagogical Institute. In practice, the most common situation is that no expert in mathematics curricula has participated in the design and political decision-making process in declaring this curriculum to be valid and binding. As can be seen immediately, there is no mention of the public opinion of the observant community, and the so-called professional public, although this is a process of general public interest. In addition, the participation of a highly professional scientific public cannot be registered in any way. This work can be seen as our contribution to international efforts (e.g. [42]) to put the problem of principled-philosophical orientation in designing mathematical curricula in the focus of the community of mathematics education researchers.

Curriculum developers need to work with education policy makers, school management, researchers in mathematics education and mathematics teachers in designing curriculum to ensure the desired required alignment of standards in mathematics education and social principled-philosophical orientations in that education (compare with [31] and [35]).

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